Lecture Notes for Differential Equations

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Contents

Chapter 1

Introduction

In this chapter, we provide a foundation for the study of differential equations. We introduce some basic concepts frequently used in this course. We also introduce some mathematical models that describes some physical processes by using differential equations.

1.1 Some Basic Mathematical Models; Direction Fields

Keywords: differential equations, mathematical models, direction fields

Many of the principles or laws underlying the behavior of the natural world are statements or relations involving rates of change at which things happen. When expressed in mathematical terms, the relations are equation and the rates of change are derivatives of some functions. To describe these physical processes quantitatively, we have to deal with equations containing derivatives of some unknown functions. These equations are called **differential equations**.

A differential equation that describes some physical process is often called a **mathematical model** of the process. In this section, we begin with a simple model leading to equation that is (relatively) easy to solve. Although our main goal is not the modeling part of physical process, it is helpful to gain motivation for the study of some (linear or nonlinear) differential equations.

Example 1.1.1 (Falling Object)**.** In this example, we are going to formulate a differential equation that describes the motion of a falling object in the atmosphere near sea level. The motion takes place during a certain time interval and we use the symbol t to denote time. We use v to represent the velocity of the falling object. The velocity will presumably change with time and we can think of v as a function of t.

The physical law that governs the motion of objects is **Newton's law**, which states that the mass of the object times its acceleration is equal to the net force on the object. In terms of mathematics, this law is expressed by the equation

$$F = ma = m\frac{dv}{dt}, \tag{1.1}$$

where m is the mass of the falling object, $a = dv/dt$ denotes its acceleration, and F is the net force exerted on the object.

Next, we consider the forces that act on the object as it falls. We have gravity that exerts a force to the object, whose magnitude is proportional to the weight of object, or mg, where g is the acceleration due to gravity. In our case, we can treat as a known physical constant. On the other hand, there is a force due to air resistance, or drag, that is more difficult to model. We assume that the magnitude of the drag force is proportional to the velocity of the object, or γv, where γ is the drag coefficient. Again, we can treat γ here as a known constant depends on the falling object itself.

In writing an expression for the net force F, since gravity always acts in downward direction, whereas, for a falling object, drag acts in the upward direction. Thus, we can write

$$F = mg - \gamma v \tag{1.2}$$

and the equation (1.1) becomes (since $m \neq 0$)

$$m\frac{dv}{dt} = mg - \gamma v \implies \frac{dv}{dt} = g - \frac{\gamma}{m}v. \tag{1.3}$$

The differential equation (1.3) is a mathematical model for the velocity v of an object falling in the atmosphere near sea level, where m, g, and γ are parameters in this model. By solving the equation (1.3), we mean to find a function $v = v(t)$ such that it satisfies the equation.

In the next section, we will show how to solve (1.3). For the present, we (qualitatively) analyze the equation (1.3) and its solution without actually finding any of them. We introduce the notion of **direction field** by continuing the example of falling object with concrete case.

Example 1.1.2 (Qualitative analysis of falling object problem)**.** We continue to study the differential equation (1.3) without solving it. To simplify our discussion, we assume that $m = 10$, $g = 9.8$ and $\gamma = 2$. Then, the equation (1.3) becomes

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{1.4}$$

In the equation above, t is the independent variable and $v = v(t)$ is a dependent variable. Note that the right-hand side $f(v,t) := 9.8 - v(t)/5$ of the equation (1.4) depends on the values of v and thus also on t. If we are given values of v and t, we can evaluate the function $f(v,t)$ with such given data. As a result, we can find the corresponding value of dv/dt.

For instance, if $v = 40$, then $dv/dt = 9.8 - 40/5 = 1.8$. This means that the slope of a solution curve $v = v(t)$ has the value 1.8 at any point where $v = 40$. We can display this information graphically in the tv-plane by drawing short line segments with slope 1.8 at several points on the line $v = 40$. Proceeding in the same way with other values of v, we create a **direction field** for the equation (1.4). A direction field for equation (1.4) is shown in Figure 1.1.

We know that a solution of equation (1.4) is a function $v = v(t)$, whose graph is a curve in the tv-plane. The importance of the direction field is that each line segment is a tangent line to one of these solution curves. Thus, even though we have not found any solutions, and no graphs of solutions appear in the figure, we can nonetheless draw some qualitative conclusions about the behavior of the solutions. For instance, if v is less than a certain critical value, say $v \leqslant 49$ for (1.4), then all the line segments have positive slopes, and the speed of the falling object increases as it falls. On the other hand, if v is greater than 49, the line segments have negative slopes, and the falling object slows down as it falls.

In fact, the constant function $v(t) = 49$ is a solution of (1.4). To verify this, substitute $v(t) = 49$ into equation and observe that each side of the equation is zero. Because it does not change with time, the solution $v(t) = 49$ is called an **equilibrium solution**. It is the solution that corresponds to a perfect balance between gravity and drag. To find out the equilibrium solution, one may solve the zero(s) of the right-hand side, namely, we solve

$$9.8 - \frac{v}{5} = 0 \implies v = 49.$$

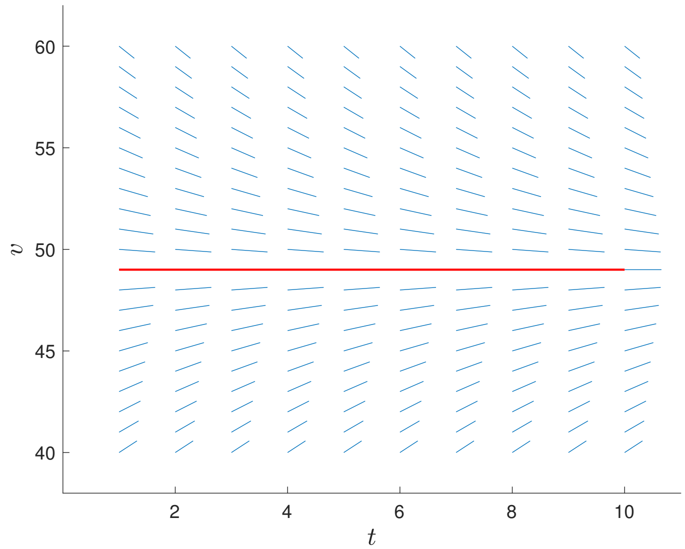


Figure 1.1: The direction field for the differential equation $v'(t) = 9.8 - v(t)/5$. Red line: $v = 49$ where $v'(t) = 0$.

In Figure 1.1, we show the equilibrium solution $v(t) = 49$ superimposed on the direction field. From the figure we can draw another conclusion, namely, that all other solutions seem to be converging to the equilibrium solution as t increase. Thus, in this context, the equilibrium solution is often called the **terminal velocity** for the falling object.

In general, direction fields are valuable tools in studying the solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y), \tag{1.5}$$

where $f(t, y)$ is a given function of the two variables t and y, sometimes referred to as the **rate function**. A direction field for equations of the form (1.5) can be drawn by evaluating f at each point of a rectangular grid. At each point of the grid, a short line segment is drawn whose slope is the value of f at that point. A direction field drawn on a fairly fine grid gives a good picture of the overall behavior of solutions of a differential equation and the construction of a direction field is often a useful first step in the investigation of a differential equation.

We summarize the content mentioned above:

- In constructing a direction field, we do not have to solve equation (1.5) and we just have to evaluate the given rate function $f(t, y)$ several times.
- Direction fields can be readily constructed even for equations quite difficult to solve.
- Drawing a direction field can be done by computer-aided procedure. For instance, one can use a MATLAB built-in function `quiver` to visualize direction field of differential equation.

1.2 Solutions of Some Differential Equations

Keywords: general solution, integral curve, initial condition, initial-value problem

In the previous section, we derived a differential equation describes the velocity of a falling object with mass m as follows:

$$\frac{dv}{dt} = g - \frac{\gamma}{m} v.$$

Once a differential equation is set up, we are not just only to analyze qualitatively by sketching direction field, but also to focus on how to solve the equation.

In this section, we introduce a simple way to solve the following differential equation

$$\frac{dy}{dt} = ay - b, \tag{1.6}$$

where a and b are given constants. We first look at the concrete setting of the problem of the falling object.

Example 1.2.1. Let us consider a falling object with acceleration of gravity $g = 9.8$, mass $m = 10$ and drag coefficient $\gamma = 2$. The equation of motion becomes

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} = \frac{49 - v}{5}. \tag{1.7}$$

Find solutions of this equation.

Solution. First, we rewrite (1.7) as follows:

$$\frac{dv/dt}{v - 49} = -\frac{1}{5}.$$

Using the chain rule, the left-hand side of the above equation can be written as

$$\frac{dv/dt}{v - 49} = \frac{d}{dt}\left[\log|v(t) - 49|\right] = -\frac{1}{5}.$$

Here, log denotes the natural logarithmic function with base $e = 2.7182818\cdots$. By integrating both sides with respect to the variable t, we obtain

$$\log|v(t) - 49| = -\frac{t}{5} + C,$$

where C is an arbitrary constant of integration. Therefore, by taking the exponential of both sides, we find that

$$|v(t) - 49| = \exp\left(-\frac{t}{5} + C\right) = e^C e^{-t/5},$$

or we can write

$$v(t) - 49 = \pm e^C e^{-t/5}$$

for any arbitrary constant C. Since C is arbitrary, so is $\pm e^C$ and we denote $c = \pm e^C$. As a result, we find that

$$v(t) = 49 + ce^{-t/5} \quad \text{for any constant } c. \tag{1.8}$$

This formula is called the **general solution** of the differential equation (1.7). The geometric representation of this general solution is an infinite family of curves called **integral curves**. Each integral curve is associated with a particular value of c and is the graph of the solution corresponding to that value of c. See Figure 1.2 for graphical illustration. □

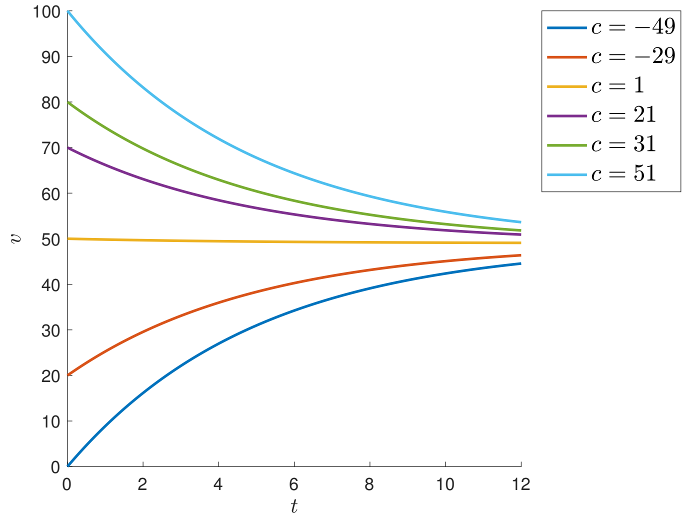


Figure 1.2: The solution $v(t) = 49 + ce^{-t/5}$ with different values of c.

In Example 1.2.1, we found infinitely many solutions of the differential equation (1.7), corresponding to the infinitely many values that the arbitrary constant c in the solution formula (1.8) might have. This is typical of what happens when one solves a differential equation. The solution process involves an integration, which brings with it an arbitrary constant, whose possible values generate a family of (infinitely many) solutions.

If we want to uniquely determine the value of c in the formula (1.7), we have to require an additional condition for the function $v(t)$. For example, if one assumes that the solution $v(t)$ satisfies

$$v(0) = 0, \tag{1.9}$$

then using the solution formula, we have

$$v(0) = 49 + ce^0 = 49 + c = 0 \implies c = -49.$$

Consequently, the solution $v(t)$ with $v(0) = 0$ is

$$v(t) = 49(1 - e^{-t/5}).$$

The assumption (1.9) is an example of an **initial condition**. We call that the differential equation (1.7) and the initial condition (1.9) forms an **initial-value problem**.

After we find the formula of the solution, we can further analyze the solution quantitatively. For instance, we can find the time it will take to fall to the ground, and the terminal speed when it hits the ground.

Example 1.2.2. We continue to study the falling object problem in Example 1.2.1. Suppose that this object is dropped from a place of height $h = 300$ with $v(0) = 0$. How long will it take to fall to the ground, and how fast will it be moving at the time of impact?

Solution. By previous results, the velocity function of the falling object with $v(0) = 0$ is

$$v(t) = 49(1 - e^{-t/5}).$$

To find the velocity of the object when it hits the ground, we need to know the time at which impact occurs. To do this, we note that the distance x the object has fallen is related to its velocity $v(t)$ by the differential equation

$$\frac{dx}{dt} = v(t) = 49(1 - e^{-t/5}).$$

Consequently, by integrating both sides with respect to t, we have

$$x(t) = 49t + 245e^{-t/5} + k,$$

where k is an arbitrary constant of integration. The object starts to fall when $t = 0$, so we know that $x = 0$ when $t = 0$ since the object does not start to fall at the beginning. Then, we have

$$0 = x(0) = 245 + k \implies k = -245$$

and we have

$$x(t) = 49t + 245e^{-t/5} - 245.$$

Let T be the time at which the object hits the ground; then $x = 300$ when $t = T$. By substituting these values in the formula above, we have

$$49T + 245e^{-T/5} - 245 = 300.$$

Solving the root of the above nonlinear system, we have $T \approx 10.51$. At this time, the corresponding velocity $v(T)$ is found from the formula and $v(T) \approx 43.01$. □

1.3 Classification of Differential Equations

Keywords: ordinary/partial differential equations, systems of differential equations, order of differential equations, linear/nonlinear differential equations, solutions of differential equations

In this section, we introduce some terminologies that are used to describe differential equations. Also, we introduce a framework of classifying differential equations. This helps us better understand and present the differential equations that we may encounter in this course and in the real world.

1.3.1 Ordinary and Partial Differential Equations

The first classification of differential equations is based on whether the unknown function depends on a single independent variable or on several independent variables. A differential equation is said to be an **ordinary differential equation** if the unknown function depends on a single independent variable and only usual derivatives with respect to this single variable appear in the differential equation. Examples of ordinary differential equation can be listed as follows:

- The equation

  $$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

  introduced in previous sections is an ordinary differential equation since there is only one independent variable t and only the derivatives with respect to t appears.

- Let $P(t), Q(t)$, and $f(t)$ be given. The equation with the form

  $$\frac{d^2y(t)}{dt} + P(t)\frac{dy(t)}{dt} + Q(t)y(t) = f(t)$$

  is also an ordinary differential equation. Besides the first-order derivative, there is also second-order derivative in the equation.

If the unknown function depends on more than one variables and partial derivatives are involved in the differential equation, then it is called a **partial differential equation**. Typical examples of partial differential equations includes:

- the heat equation
$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t};$$
- the wave equation
$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$
since the unknown function $u(x,t)$ depends on two differential independent variables x and t, and partial derivatives with respect to them appear in the differential equation.

In this course, we only focus on ordinary differential equations.

1.3.2 System of Differential Equations

One can classify differential equations depends on the number of unknown functions that are involved. If there is a single function to be determined, then one differential equation is sufficient. However, if there are more than one unknown functions, then a system of differential equations is required. The number of unknown functions should be consistent with the number of the equations in the system.

Example 1.3.1. The Lotka-Volterra equations are important in ecological modeling. They have the form

$$\frac{dx}{dt} = ax - \alpha xy,$$
$$\frac{dy}{dt} = -cy + \gamma xy,$$

where $x(t)$ and $y(t)$ are the unknown functions. Each of them depends on a single independent variable t. Here, a, α, c, and γ are some empirical constants. This is a system of differential equations with two unknown functions and two (differential) equations.

The theory of system of equations are discussed Chapters 7 and 9. In some areas of application, it is not unusual to encounter very large systems containing hundreds, or even many thousands, of differential equations.

1.3.3 Order of Differential Equations

Another classification of differential equations depends on the order of the highest derivatives that appears in the equation. The highest order of derivation that appears in a differential equation is called the **order** of the differential equation. For example, the following differential equation

$$\frac{dy}{dt} = 9.8 - \frac{y}{5}$$

is a first-order (ordinary) differential equation (or the equation is of order 1); the equation below

$$\frac{d^2y(t)}{dt} + P(t)\frac{dy(t)}{dt} + Q(t)y(t) = f(t)$$

is of second-order. More generally, the equation of the form

$$F\left(t, y(t), y'(t), \cdots, y^{(n)}(t)\right) = 0 \tag{1.10}$$

is an ordinary differential equation of order n. This expresses a relation between the independent variable t and the values of the function $y(t)$ and its first n derivatives. Sometimes we write y for $y(t)$ with $y^{(n)}$ standing for $y^{(n)}(t)$ for any positive integer n. For example, the following equation

$$y''' + 2e^t y'' + yy' = t^4$$

is a third-order differential equation for the unknown function $y = y(t)$. Occasionally, other symbols can be used instead of t and y for the independent and dependent variables; the meaning should be clear from the context.

1.3.4 Linear and Nonlinear Equations

A crucial classification of differential equation is whether they are linear or nonlinear. A differential equation is said to be **linear** if it is defined by a linear polynomial in the unknown functions and its derivatives. That is, the general linear differential equation of order n is of the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t), \tag{1.11}$$

where $a_0(t), \cdots, a_n(t)$ and $g(t)$ are arbitrary differentiable functions. A differential equation that is not of the form (1.11) is called a **nonlinear** differential equation.

In most of the applications, the word *linear* generally means "simple" and nonlinear means "complicated". The theory for solving linear equations is very well developed because linear equations are simple enough to be solvable. Nonlinear equations can usually not be solved exactly and are the subject of much ongoing research. In our course, we focus on the theory of solving linear differential equation. We will slightly discuss the qualitative theory of nonlinear differential equation in Chapter 9.

Here are some examples of linear and nonlinear equations. We assume that the unknown function is $y = y(t)$.

- The equation
  $$y'' + y = 0$$
  is a (second-order, ordinary) linear differential equation.
- The equation
  $$2y' + e^t y = \cos(t)$$
  is a (first-order, ordinary) linear differential equation with $n = 2$ in the general form of linear differential equation (1.11) ($a_0(t) = 2$, $a_1(t) = e^t$, and $g(t) = \cos(t)$).
- The equation
  $$y' + \frac{1}{y} = 0$$
  is a (first-order, ordinary) nonlinear equation since the term $1/y$ is not one of the term in (1.11).
- The equation
  $$y'' + \frac{g}{L}\sin y = 0$$
  describes the oscillation of a pendulum. It is a (second-order, ordinary) nonlinear equation since the term $(g/L)\sin y$ is not one of the term in (1.11).

1.3.5 Solutions of differential equations

In this subsection, we give a formal definition of solutions of differential equations. We say that the function $\phi = \phi(t)$ is a **solution** of the following n-th order ordinary equation

$$F(t, y, y', \cdots, y^{(n)}) = 0$$

on the interval $\alpha < t < \beta$ if ϕ is continuous in the interval (α, β) and $\phi', \phi'', \cdots, \phi^{(n)}$ exist and satisfy the n-th order differential equation. That is,

$$F(t, \phi, \phi', \cdots, \phi^{(n)}) = 0$$

for every $t \in (\alpha, \beta)$.

It is often not so easy to find solutions of differential equations. However, if you find a function that you think may be a function of a given equation, it is usually relatively easy to determine whether the function is actually a solution: just substitute the function into the equation.

Example 1.3.2. It is easy to show that the function $y_1(t) = \cos t$ is a solution of the following second-order (ordinary) differential equation:

$$y'' + y = 0 \tag{1.12}$$

for all t. To confirm this, observe that $y_1'(t) = -\sin t$ and $y_1''(t) = -\cos t$; then it follows that $y_1''(t) + y_1(t) = 0$ for all t. In the same way it is easy to verify that $y_2(t) = \sin t$ is also a solution of (1.12).

1.4 Additional Reading: Existence and Uniqueness of Differential Equations

Keywords: existence/uniqueness of solutions

Although for the case of (1.12) we are able to verify that certain simple functions are solutions, in general we do not have such solutions readily available. Thus, a fundamental question is the following: Does a differential equation always have a solution? The answer is “No”. Merely writing down a differential equation does not necessarily mean that there is a function that satisfies it. This is the question of **existence** of a solution, and it is answered by theorems stating that under certain conditions, the equation always has solutions. This is not a purely theoretical concern for at least two reasons. If a problem has no solution, we would prefer to know that fact before investing time and effort in a vain attempt to solve the problem. Further, if a sensible physical problem is modeled mathematical as a differential equation, then the equation should have a solution. If it does not, then presumably there is something wrong with the formulation. In this sense an engineer or scientist has some check on the validity of the mathematical model.

If we know already that a given differential equation has at least one solution, then we may need to consider how many solutions it has, and what additional conditions must be specified to single out a particular solution. This is the question of **uniqueness**. In general, solutions of differential equations contain one or more arbitrary constant of integration. As in the question of existence of solutions, the issue of uniqueness has practical as well as theoretical implications. If we are fortunate enough to find a solution of a given problem, and if we know that the problem has a unique solution, then we can be sure that we have completely solved the problem. If there may be other solutions, then perhaps we should continue to search for them.

After imposing these two issues, the third important question is: Given a differential equation, can we actually determine a solution, and if so, how? Note that if we find a solution of the given equation, we have at the same time answered the question of the existence of a solution. However, without knowledge of existence theory we might, for example, use a computer to find a numerical approximation to a "solution" that does not exist. On the other hand, even though we may know that a solution exists, it may be hat the solution is not expressible in terms of the usual elementary functions. Unfortunately, this is the situation for most differential equations. Thus, we discuss both elementary methods that can be used to obtain exact solutions of certain relatively simple problems, and also methods of a more general nature that can be used to find approximations to solution of more difficult problems.

1.5 Exercises

There are 4 questions in this assignment. Answer all. Please submit your homework on **Gradescope**. The deadline is **5:00 pm (CDT), Sep 9 2022**.

1. (5 points) Consider the following differential equation with unknown function $p = p(t)$:

$$\frac{dp}{dt} = \frac{p}{3} - 50. \tag{1.13}$$

   (a) Sketch the direction field of (1.13) (by hand or any programming language).

   (b) Find out the equilibrium solution of (1.13). Plot the equilibrium solution in the same frame of the direction field.

   (c) From the graph of the direction field plotted in (a), does a solution of (1.13) always converge to the equilibrium solution found in (b)? Explain briefly.

2. (5 points) Consider the following differential equation with unknown function $y = y(t)$:

$$\frac{dy}{dt} = ay - b \tag{1.14}$$

   where a and b are two given constants and $a \neq 0$.

   (a) Using the technique in Example 1.2.1, find the general solution of the equation (1.14).

   (b) If the unknown function $y = y(t)$ satisfies the initial condition $y(0) = y_0$, where y_0 is a given constant such that $ay_0 - b \neq 0$. Express the solution $y(t)$ in terms of a, b, and y_0.

   (c) If $a = 0$, what is the general solution of (1.14)?

3. (5 points) Determine the order of the given differential equations and also state whether the equation is linear or nonlinear. Assume that the unknown function is $y = y(t)$.

   (a) $t^2 y'' + ty' + 2y = \sin t$.

   (b) $y'' + \sin(t + y) = \cos t$.

   (c) $y^{(4)} + y''' + y'' + y' + y = 1$.

4. (5 points) For each differential equation stated below, determine the value(s) of r for which the given differential equation has solutions of the form $y = e^{rt}$.

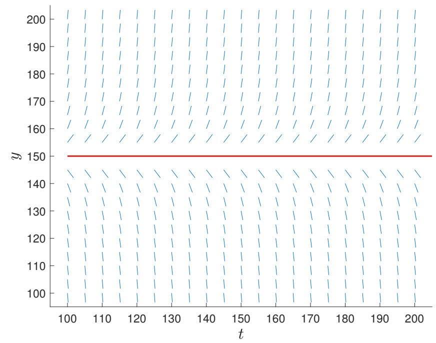
   (a) $y' + 2y = 0$.

   (b) $y'' + y' - 6y = 0$.

   (c) $y''' - 3y'' + 2y' = 0$.

Reference Solutions of Exercises in Chapter 1

1. The direction field is shown in the figure below. The equilibrium solution $p = 150$ is plotted in red. The solution may not converge to $p = 150$ since it is unstable.



2. (a) One can solve by the method of separating variables. Assume that $a \neq 0$. We have

$$\begin{aligned}
\frac{dy}{dt} &= ay - b \\
\frac{1}{a}\frac{dy}{y - \frac{b}{a}} &= dt \\
\frac{1}{a}\log\left|y - \frac{b}{a}\right| &= t + C \\
\log\left|y - \frac{b}{a}\right| &= at + C \\
\left|y - \frac{b}{a}\right| &= Ce^{at} \\
y(t) &= \frac{b}{a} + Ce^{at}
\end{aligned}$$

with C being arbitrary.

(b) Given initial condition $y(0) = y_0$, we find the value of C by plugging in $t = 0$ and $y = y_0$ in the above general solution and obtain

$$C + \frac{b}{a} = y_0 \implies C = y_0 - \frac{b}{a}.$$

The solution $y(t)$ becomes

$$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at} = y_0 e^{at} + \frac{b}{a}\left(1 - e^{at}\right).$$

(c) If $a = 0$, then the differential equation becomes $y' = -b$ and $y(t) = -bt + C$ will be the general solution.

3. (a) Second order; linear.
   (b) Second order; nonlinear.
   (c) Forth order; linear.

4. (a) $(r + 2)e^{rt} = 0 \implies r = -2$.
   (b) $(r^2 + r - 6)e^{rt} = 0 \implies (r + 3)(r - 2) = 0 \implies r = -3$ or $r = 2$.
   (c) $(r^3 - 3r^2 + 2r)e^{rt} = 0 \implies r(r^2 - 3r + 2) = 0 \implies r(r - 1)(r - 2) = 0 \implies r = 0$, or $r = 1$, or $r = 2$.

Chapter 2

First-Order Differential Equations

This chapter deals with differential equation of first order:

$$\frac{dy}{dt} = f(t, y) \tag{2.1}$$

where f is a given function of two variables and the unknown function is $y = y(t)$. The objective is to determine whether a solution exists and if so, to develop methods for finding solutions. For an arbitrary function f, there is no general method for solving the equation in terms of elementary functions .Instead, we will describe several methods, each of which is applicable to certain subclass of first-order equations.

The most important of these are linear equations (Section 2.1), separable equations (Section 2.2), and exact equations (Section 2.6). Other sections of this chapter describe some of the important applications of first-order differential equations.

2.1 Linear Differential Equations; Method of Integrating Factors

Keywords: integrating factors, product rule, linear differential equations

If the function f in (2.1) depends linearly on the dependent variable y, then (2.1) is a first-order linear differential equation. A typical example is

$$\frac{dy}{dt} = -ay + b$$

where a and b are given constants. Recall that an equation oft his form describes the motion of an object falling in the atmosphere.

We consider a more general first-order linear differential equation in the standard form

$$\frac{dy}{dt} + p(t)y = g(t), \tag{2.2}$$

where $p(t)$ and $g(t)$ are given functions of the independent variable t. Sometimes it is more convenient to write the equation in the form

$$P(t)\frac{dy}{dt} + Q(t)y = G(t), \tag{2.3}$$

where $P(t)$, $Q(t)$, and $G(t)$ are given such that $P(t) \neq 0$ for some t. One can convert (2.3) to (2.2) by dividing both sides of equation (2.3) by $P(t)$ if $P(t) \neq 0$ for some t.

In some cases it is possible to solve a first-order linear differential equation immediately by integrating the equation, as in the next example.

Example 2.1.1. Solve the differential equation

$$(4+t^2)\frac{dy}{dt}+2ty=4t. \tag{2.4}$$

Solution. Note that using the product rule, we have

$$(4+t^2)\frac{dy}{dt}+2ty=\frac{d}{dt}\left((4+t^2)y\right).$$

It follows that the equation (2.4) can be rewritten as

$$\frac{d}{dt}\left((4+t^2)y\right)=4t.$$

Integrating both sides with respect to t, have

$$(4+t^2)y=2t^2+C$$

where C is an arbitrary constant of integration. Solving for y, we have

$$y=\frac{2t^2+C}{4+t^2}$$

with C being arbitrary constant. This is the general solution of (2.4). □

Most first-order linear differential equation cannot be solved as in Example 2.1.1 because their left-hand sides are not the derivative of the product of y and some other function. However, if the differential equation is multiplied by a certain function denoted by $\mu(t)$, then the equation is converted into one that is immediately integrable by using the product rule for derivatives just as in Example 2.1.1. The function $\mu(t)$ is called an **integrating factor** and our task in this section is to determine how to find it for a given equation. We show how this method works first for a concrete example and then for the general case in the standard form (2.2).

Example 2.1.2. Find the general solution of the following differential equation

$$\frac{dy}{dt}+\frac{1}{2}y=\frac{1}{2}e^{t/3}. \tag{2.5}$$

Also, find the particular solution such that $y(0)=1$.

Solution. The first step is to multiply equation (2.5) by a function $\mu(t)$ (to be determined) such that

$$\mu(t)\frac{dy}{dt}+\frac{1}{2}\mu(t)y=\frac{d}{dt}\left(\mu(t)y\right)=\mu(t)\frac{dy}{dt}+\frac{d\mu(t)}{dt}y. \tag{2.6}$$

The second inequality comes from the product rule. If this is the case, then equation (2.5) can be rewritten as

$$\frac{d}{dt}\left(\mu(t)y\right)=\frac{1}{2}\mu(t)e^{t/3}$$

and it can be solved by integrating both sides of the equation above with respect to t. Thus, from (2.6), the integrating factor μ should satisfy

$$\frac{d\mu(t)}{dt}=\frac{1}{2}\mu(t).$$

Our search for an integrating factor will be successful if we can find μ such that it satisfies the equation above. In fact, the function $\mu(t)$ given by $\mu(t) = ce^{t/2}$ (for any constant c) is an integrating factor for equation (2.5). Since we do not need the most general integrating factor, we can choose $c = 1$ and use $\mu(t) = e^{t/2}$. Now we return to solve (2.5). We obtain

$$\frac{d}{dt}(e^{t/2}y) = \frac{1}{2}e^{t/2+t/3} = \frac{1}{2}e^{5t/6}.$$

By integrating both sides with respect to t, we obtain

$$e^{t/2}y = \frac{1}{2}\cdot\frac{6}{5}e^{5t/6} + c = \frac{3}{5}e^{5t/6} + c$$

for any arbitrary constant c. Finally, dividing both sides by $e^{t/2}$ we have the general solution for (2.5), namely,

$$y(t) = \frac{3}{5}e^{t/3} + ce^{-t/2}.$$

To find the solution satisfying $y(0) = 1$, we let $t = 0$ in the general solution and we have

$$1 = y(0) = \frac{3}{5} + c \implies c = \frac{2}{5}.$$

The desired solution is

$$y(t) = \frac{3}{5}e^{t/3} + \frac{2}{5}e^{-t/2}.$$

□

Now we turn to the general first-order linear differential equation (2.2). To determine an approximate integrating factor, we multiply (2.2) by an to-be-determined function $\mu(t)$, and we obtain

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t). \tag{2.7}$$

Following the same line of development as in Example 2.1.2, we see that the left-hand side of the above equation is the derivative of the product $\mu(t)y$, provided that $\mu(t)$ satisfies the following condition:

$$\frac{d\mu(t)}{dt} = p(t)\mu(t).$$

If we assume that $\mu(t)$ is positive, then we have

$$\frac{1}{\mu(t)}\frac{d\mu(t)}{dt} = p(t),$$

and consequently

$$\log|\mu(t)| = \int p(t)\ dt + k$$

for any arbitrary constant k of integration. By choosing $k = 0$, we obtain the simplest possible integrating factor μ, namely,

$$\mu(t) = \exp\left(\int p(t)\ dt\right). \tag{2.8}$$

Note that $\mu(t)$ is positive for all t, as we assumed. Returning to (2.7), we have

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t),$$
$$\implies \mu(t)y = \int \mu(t)g(t)\ dt + C,$$

where C is an arbitrary constant. Sometimes the integral in the above equation can be evaluated in terms of elementary functions. However, in general this may not be possible, so the general solution of (2.2) is

$$y(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^{t} \mu(s)g(s)\ ds + C \right), \tag{2.9}$$

where again t_0 is some convenient lower limit of integration. Observe that the general solution involves two integrations, one to obtain $\mu(t)$ from (2.8) and the other to determine y from (2.9).

Example 2.1.3. Solve the initial-value problem

$$\begin{aligned} \frac{dy}{dt} + \frac{2}{t}y &= 4t, \\ y(1) &= 2, \end{aligned} \tag{2.10}$$

using the method of integrating factor.

Solution. First, we apply the method of integrating factor to find the solution of the linear differential equation in (2.10). The differential equation is already in the standard form like (2.2) with $p(t) = 2/t$ and $g(t) = 4t$ (if not, we have to convert the differential equation in standard form before using the method of integrating factor).

We may choose the integrating factor as shown in (2.8). In this case, we have

$$\mu(t) = \exp\left(\int p(t)\ dt \right) = \exp\left(\int \frac{2}{t}\ dt \right) = \exp(2\log|t|) = t^2.$$

Then, we have

$$\frac{d}{dt}\left(t^2 y\right) = t^2(4t) = 4t^3 \implies t^2 y = \int 4t^3\ dt + C = t^4 + C$$

where C is an arbitrary constant of integration. The general solution of the differential equation in (2.10) is

$$y(t) = t^2 + \frac{C}{t^2}, \quad t \neq 0$$

for any constant C. Note that for any constant $C \neq 0$, the general solution becomes unbounded and is asymptotic to the y-axis as $t \to 0$ from the right. This is the effect of the infinite discontinuity in the coefficient $p(t) = 2/t$ at the origin.

To determine the value of constant using the initial condition $y(1) = 2$, we set $t = 1$ and $y = 2$ in the general solution and we have

$$2 = y(1) = 1 + \frac{C}{1} = 1 + C \implies C = 1.$$

The solution is only valid when $t > 0$ and we have to restrict the solution to the interval $0 < t < \infty$. It is important to note that while the function $y = t^2 + 1/t^2$ for $t < 0$ is part of the general solution of the differential equation, it is not part of the solution of this initial-value problem. This is the first example in which the solution fails to exist for some values of t.

We remark that if the initial condition is $y(1) = 1$, then $C = 0$ and the solution is $y(t) = t^2$, which is bounded and differentiable even at $t = 0$. □

Example 2.1.4. Solve the initial-value problem

$$2y' + ty = t, \quad y(0) = 2. \tag{2.11}$$

Solution. We can rewrite the equation as

$$y'(t) + \frac{t}{2}y = \frac{t}{2}.$$

The integrating factor in this case is

$$\mu(t) = \exp\left(\int \frac{t}{2}\, dt\right) = e^{t^2/4}.$$

Hence, the solution is

$$y(t) = e^{-t^2/4}\left(\int \frac{t}{2}e^{t^2/4}\, dt + C\right) = Ce^{-t^2/4} + 1.$$

Using the initial condition $y(0) = 2$, we have $C + 1 = 2 \implies C = 1$. Hence, the solution is

$$y(t) = e^{-t^2/4} + 1.$$

□

2.2 Separable Differential Equations

Keywords: separable equation, Chain Rule, first-order nonlinear, implicit function

In this section, we present a technique to solve a special class of first-order differential equations using the process of direction integration. A large class of nonlinear differential equations falls into the category of the so-called **separable equations**.

We consider a subclass of first-order equations that can be solved by direction integration:

$$\frac{dy}{dt} = \frac{M(x)}{N(y)} \tag{2.12}$$

for some function $M(x)$ (depending only on x) and $N(y) \neq 0$ (depending only on y). Of course, we can rewrite the first-order differential equation (2.12) as follows:

$$N(y)\frac{dy}{dx} = M(x) \quad \text{or} \quad -M(x) + N(y)\frac{dy}{dx} = 0. \tag{2.13}$$

Such an equation is said to be **separable**. A separable equation can be solved by integrating the functions $M(x)$ and $N(y)$. We illustrate the process by an example and discuss it in general for (2.13).

Example 2.2.1. Find an equation for its integral curves for the following separable equation

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2} \tag{2.14}$$

Solution. We set $M(x) = -x^2$ and $N(y) = 1 - y^2$ in (2.14) and we can rewrite:

$$-x^2 + (1 - y^2)\frac{dy}{dx} = 0. \tag{2.15}$$

Using the Chain Rule, if y is a function of x, we have

$$\frac{d}{dx}\left(y - \frac{y^3}{3}\right) = (1 - y^2)\frac{dy}{dx}.$$

Thus, the second term in (2.15) is the derivative of $y-y^3/3$ with respect to x, and the first term is the derivative of $-x^3/3$. Thus, we can rewrite (2.15) further as:

$$\frac{d}{dx}\left(-\frac{x^3}{3}\right)+\frac{d}{dx}\left(y-\frac{y^3}{3}\right)=0 \iff \frac{d}{dx}\left(-\frac{x^3}{3}+y-\frac{y^3}{3}\right)=0.$$

Therefore, by integrating both sides with respect to x, we have

$$-\frac{x^3}{3}+y-\frac{y^3}{3}=C \tag{2.16}$$

for any constant C. The equation (2.16) is an equation for the integral curves of the differential equation (2.14). Any differentiable function $y=\phi(x)$ that satisfies (2.16) is a solution of (2.14). □

Essential the same procedure in the previous example can be followed for any separable equation. Returning to (2.13), let H_1 and H_2 be any antiderivatives of M and N, respectively. That is, we have

$$\frac{dH_1(x)}{dx}=M(x) \quad \text{and} \quad \frac{dH_2(y)}{dy}=N(y),$$

and (2.13) becomes

$$-\frac{dH_1(x)}{dx}+\frac{dH_2(y)}{dy}\frac{dy}{dx}=0.$$

If y is regarded as a function of x, then according to the Chain Rule, we have

$$\frac{dH_2(y)}{dy}\cdot\frac{dy}{dx}=\frac{d}{dx}H_2(y).$$

Consequently, we can write (2.13) as

$$\frac{d}{dx}\left(-H_1(x)+H_2(y)\right)=0.$$

Integrating both sides with respect to x, we obtain

$$-H_1(x)+H_2(y)=C \quad \text{or} \quad H_2(y)=H_1(x)+C \tag{2.17}$$

where C is an arbitrary constant. Any differentiable function $y=\phi(x)$ that satisfies (2.17) is a solution of (2.13). In other words, the formula (2.17) defines the solution implicitly rather than explicitly. In practice, the separable equation can be solved in the following *informal* manner: starting from the separable equation (2.12), we proceed

$$\begin{aligned}
&\frac{dy}{dx}=\frac{M(x)}{N(y)}\\
\Longrightarrow\ & N(y)dy=M(x)dx \quad \text{(separate the variables } x \text{ and } y)\\
\Longrightarrow\ & \int N(y)dy=\int M(x)dx \quad \text{(integrate both sides with respect to } x \text{ and } y \text{ respectively)}\\
\Longrightarrow\ & H_2(y)=H_1(x)+C
\end{aligned}$$

where C is arbitrary constant from integration. This approach is (usually) called the **method of separating variables**. The differential equation (2.13), together with an initial condition $y(x_0)=y_0$ forms an initial-value problem. To solve the initial-value problem, we determine the appropriate value for the constant C in (2.17). We do this by setting $x = x_0$ and $y = y_0$ in (2.17) and noting that

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)\ ds, \quad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)\ ds,$$

we obtain

$$\int_{x_0}^{x} M(s)\ ds + \int_{y_0}^{y} N(s)\ ds = 0. \tag{2.18}$$

The equation above is an implicit representation of the solution of the equation (2.13) that also satisfies the initial condition $y(x_0) = y_0$. Bear in mind that to determine an explicit formula for the solution, one needs to solve (2.18) for y as a function of x. Unfortunately, it is often impossible to do this analytically; in such cases you can resort to numerical methods to find approximate values of y for given values of x.

Example 2.2.2. Solve the initial-value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \quad y(0) = -1, \tag{2.19}$$

and determine the interval in which the solution exists.

Solution. The differential equation can be rewritten as

$$2(y-1)dy = (3x^2 + 4x + 2)dx.$$

Integrating the left-hand side with respect to y and the right-hand side with respect to x gives

$$y^2 - 2y = x^3 + 2x^2 + 2x + C, \tag{2.20}$$

where C is an arbitrary constant. To determine the solution satisfying the prescribed initial condition, we substitute $x = 0$ and $y = -1$ in (2.20) to obtain

$$(-1)^2 - 2(-1) = C \implies C = 3.$$

Hence, the solution of the initial-value problem is given implicitly by

$$y^2 - 2y = x^3 + 2x^2 + 2x = 3. \tag{2.21}$$

To obtain the solution explicitly, we have to solve (2.21) for y in terms of x. This is a simple matter in this case, since (2.21) is quadratic in y, and we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

It gives two solutions of differential equation, only one of which, however, satisfies the given initial condition. For instance, we define $y_1(x)$ such that

$$y_1(x) = 1 + \sqrt{x^3 + 2x^2 + 2x + 4} \implies y_1(0) = 1 + \sqrt{4} = 3 \neq -1,$$

where y_1 does not satisfy the initial condition. On the other hand, the function $y_2(x)$ defined as

$$y_2(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \implies y_2(0) = 1 - \sqrt{4} = -1,$$

satisfies the initial condition. So we finally obtain

$$y = y_2(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \tag{2.22}$$

as the solution of (2.19). To determine the interval in which the solution (2.22) is valid, we find the interval in which the quantity under the radical is positive. Notice that

$$x^3 + 2x^2 + 2x + 4 = x^2(x+2) + 2(x+2) = (x+2)(x^2+2).$$

The only real zero of this expression is $x = -2$, so the desired interval is $x > -2$. We remark that the interval of validity does not include the point $x = -2$ (even though $y(-2) = 1$), since the derivative y' is not defined when $x = -2$ and $y = 1$. □

Example 2.2.3. Solve the initial-value problem

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}, \quad y(0) = 1 \tag{2.23}$$

and determine its interval of validity.

Solution. Rewriting equation (2.23) as

$$(4 + y^3)dy = (4x - x^3)dx,$$

integrating each side, we obtain

$$4y + \frac{y^4}{4} = 2x^2 - \frac{x^4}{4} + C$$

for any arbitrary constant C. Multiplying by 4 and rearranging the terms, we obtain

$$y^4 + 16y + x^4 - 8x^2 = C$$

for any arbitrary constant C. Any differentiable function $y = \phi(x)$ that satisfies the equation above is a solution of (2.23). To find the particular solution satisfying $y(0) = 1$, we set $x = 0$ and $y = 1$ and it results in $C = 17$. Thus, the solution in question is given implicitly by

$$y^4 + 16y + x^4 - 8x^2 = 17. \tag{2.24}$$

The interval of validity of this solution extends on either side of the initial point $(x, y) = (0, 1)$ as long as the function remains differentiable. The interval ends when we reach points where the tangent line is vertical. It follows from the differential equation (2.23) that these are points where $4 + y^3 = 0$ or

$$y = (-4)^{1/3} \approx -1.5874.$$

From the solution formula (2.24), the corresponding values of x are $x = \pm 3.3488$. Hence, the interval of validity is $(-3.3488, 3.3488)$ (roughly). □

Example 2.2.4. Solve the initial-value problem

$$\frac{dy}{dx} = 2y^2 + xy^2, \quad y(0) = 1. \tag{2.25}$$

Determine where the solution attains its minimum value.

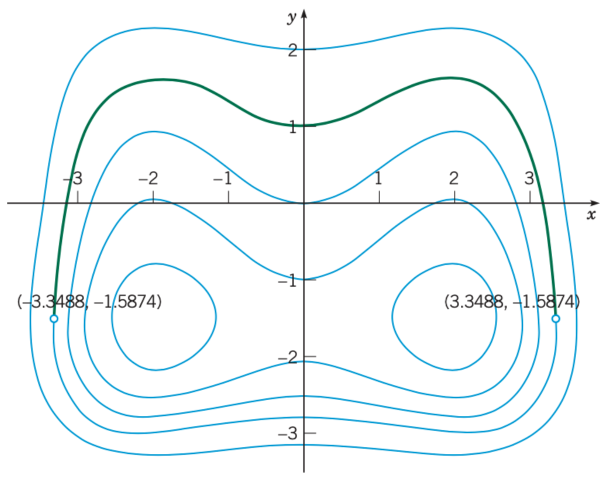


Figure 2.1: Integral curves for Example 2.2.3.

Solution. We rewrite the equation as follows:

$$\frac{dy}{dx} = y^2(x+2) \implies \frac{dy}{y^2} = (x+2)dx.$$

Integrating both sides, we obtain

$$-\frac{1}{y} = \frac{x^2}{2} + 2x + C.$$

Using the initial condition $y(0) = 1$, we obtain that

$$-1 = 0 + C \implies C = -1.$$

The solution to the initial-value problem is

$$y(x) = \left(-2x - \frac{x^2}{2} + 1\right)^{-1} = -\frac{2}{x^2+4x-2} = -\frac{2}{(x+2)^2-6}.$$

It is clear that the solution attains its minimum value at $x = -2$ with $y(-2) = \frac{1}{3}$. □

2.3 Modeling with First-Order Differential Equations

Keywords: mathematical modeling, compound interest, escape velocity, SIR model

In this section, we present some examples of mathematical modeling with first-order (system of) differential equations. The first example is related to compound interest and the second one discusses the escape velocity. Finally, we briefly present a system of differential equations that describes a basic compartmental model in epidemiology.

Example 2.3.1 (Compound interest)**.** Suppose that a sum of money, S_0, is deposited in a bank or money fund that pays interest at an annual rate r. The value $S(t)$ of the investment at any time t depends on the frequency with which interest is compounded as well as on the interest rate. Assume that compounding takes place continuously. We would like to set up an initial-value problem that describes the growth of the investment.

The rate of change of the value of the investment is dS/dt, and this quantity is equal to the rate at which interest accrues, which is the interest rate r times the current value of investment $S(t)$. Thus, we have

$$\frac{dS}{dt} = rS. \tag{2.26}$$

It is the differential equation that governs the process. Let t denote the times in years. Since the initial value S_0 is deposited, the corresponding initial condition is

$$S(0) = S_0.$$

The solution of this initial-value problem gives the balance $S(t)$ in the account at any time t. This initial-value problem is readily solved, since the differential equation (2.26) is both linear and separable. Consequently, we find that

$$S(t) = S_0 e^{rt}.$$

Thus, a bank account with continuously compounding interest grows exponentially. The model is easily extended to situations involving deposits or withdrawals in addition to the accrual of interest, dividends, or annual capital gains. If we assume that the deposits or withdrawals take place at a constant rate k, then (2.26) is replaced by

$$\frac{dS}{dt} = rS + k \quad \text{or} \quad \frac{dS}{dt} - rS = k, \tag{2.27}$$

where k is positive for deposit and negative for withdrawals. The equation is linear with integrating factor $\mu(t) = e^{-rt}$, and thus we can solve (2.27) using method of integrating factors. Thus, we can find out the general solution:

$$\begin{aligned} e^{-rt}\frac{dS}{dt} - e^{-rt} rS &= ke^{-rt} \\ \frac{d}{dt}\left(e^{-rt}S\right) &= ke^{-rt} \\ e^{-rt}S &= -\frac{k}{r}e^{-rt} + C \\ \implies S(t) &= Ce^{rt} - \frac{k}{r}. \end{aligned}$$

If we consider the initial condition $S(0) = S_0$, then we have

$$S(0) = C - \frac{k}{r} = S_0 \implies C = S_0 + \frac{k}{r}.$$

The solution is

$$S(t) = \left(S_0 + \frac{k}{r}\right)e^{rt} - \frac{k}{r} = S_0 e^{rt} + \frac{k}{r}(e^{rt} - 1). \tag{2.28}$$

Example 2.3.2 (Escape velocity)**.** A body of constant mass m is projected away from the earth in a direction perpendicular to the earth's surface with an initial velocity v_0. Assuming that there is no air resistance, but taking into account the variation of the earth's gravitational field with distance.

We would like to find an expression for the velocity during the ensuing motion. Also, we will find the initial velocity that is required to lift the body to a given maximum altitude $A_{\max}$ above

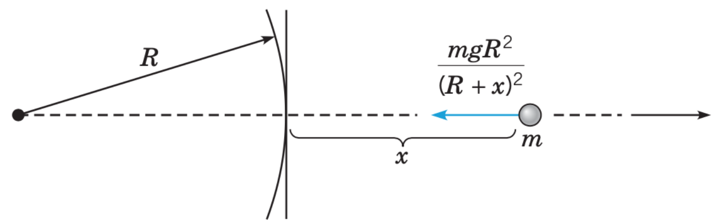


Figure 2.2: A body in the earth's gravitational field is pulled towards the center of the earth.

the surface of the earth, and find the least initial velocity for which the body will not return to the earth; the latter is the **escape velocity**.

Let the positive x-axis point away from the center of the earth along the line of motion with $x = 0$ lying on the earth's surface. Denote R is the radius of the earth and g the acceleration due to gravity. The differential equation that describes the velocity is

$$v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}. \tag{2.29}$$

The initial condition is $v(0) = v_0$. The equation (2.29) is separable but not linear. Integrating both sides with respect to v and x respectively, we obtain

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + C$$

where C is an arbitrary constant. Making use of the initial condition, we have

$$\frac{v_0^2}{2} = gR + C \implies C = \frac{v_0^2}{2} - gR$$

and the solution becomes

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + \frac{v_0^2}{2} - gR \implies v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}. \tag{2.30}$$

Note that equation (2.30) gives the velocity as a function of altitude rather than as a function of time. The plus sign must be chosen if the body is rising, and the minus sign must be chosen if it is falling back to earth. To determine the maximum altitude $A_{\max}$ that the body reaches, we set $v = 0$ and $x = A_{\max}$ in (2.30) and then solve for $A_{\max}$, obtaining

$$A_{\max} = \frac{v_0^2 R}{2gR - v_0^2}.$$

Solving for v_0, we find the initial velocity required to lift the body to the altitude $A_{\max}$, namely

$$v_0 = \sqrt{2gR\frac{A_{\max}}{R + A_{\max}}}.$$

The escape velocity v_e is then found by letting $A_{\max} \to \infty$. Consequently, we have

$$v_e = \sqrt{2gR}.$$

We remark that the numerical value of v_e is approximately 6.9 mi/s, or 11.1 km/s.

Example 2.3.3 (Compartmental model in epidemiology)**.** The susceptible-infectious-recovered (SIR) model is one of the simplest compartmental models. Many epidemiological models are derivatives of this basic form of the SIR model. The model consists of three compartments:

S: the number of susceptible individuals. When a susceptible and an infectious individual come into *infectious contact*, the susceptible individual contracts the disease and transitions to the infectious compartment.

I: the number of infectious individuals. These are individuals who have been infected and are capable of infecting susceptible individuals.

R: the number of removed (and immune) or deceased individuals. These are individuals who have been infected and have either recovered from the disease and entered the removed compartment, or died. It is assumed that the number of deaths is negligible with respect to the total population. This compartment may also be called *recovered* or *resistant*.

The variables S, I, and R represent the fractions of persons in each compartment at a particular time. To represent that the number of susceptible, infectious and removed individuals may vary over time (even if the total population size remains constant), we make the precise numbers a function of t (time): $S(t)$, $I(t)$ and $R(t)$. For a specific disease (e.g. COVID-19 or flu) in a specific population, these functions may be worked out in order to predict possible outbreaks and bring them under control.

Since $S(t)$, $I(t)$, and $R(t)$ represent the susceptible, infected, and recovered/removed fractions of persons involved in the infection at time t, so we have

$$S(t) + I(t) + R(t) = 1 \tag{2.31}$$

for any time t, and because they are fractions, they must all reside within the interval $[0, 1]$. If we denote $a(t)$ and $\mu(t)$ the semi-positive infection and recover rates, respectively, the SIR-model is defined with the following first-order (nonlinear) system:

$$\begin{aligned} \frac{dS}{dt} &= -a(t)SI, \\ \frac{dI}{dt} &= a(t)SI - \mu(t)I, \\ \frac{dR}{dt} &= \mu(t)I. \end{aligned} \tag{2.32}$$

So far we may not be able to find the solution of this SIR model analytically. We will conduct some qualitative analysis to understand better this model. The second equation

$$\frac{dI}{dt} = a(t)SI - \mu(t)I,$$

describes the rate of change of the portion of people get infected by a specific disease at any time t. For the time being, we assume that $a(t) = a$ and $\mu(t) = \mu$ are just constant functions. If we think about whether there is going to be a outbreak of pandemic, we investigate the equation at the beginning, namely, we take $t = 0$ and obtain

$$\left.\frac{dI}{dt}\right|_{t=0} = aS(0)I(0) - \mu I(0).$$

Here, $S(0)$ and $I(0)$ denotes the initial fractions of people which are susceptible and infected, respectively. We may assume that $I(0) > 0$ initially (if $I(0) = 0$ then there is no infected people and no outbreak happens). If the rate of change is negative, that means the number of infected people will not grow and

$$\left.\frac{dI}{dt}\right|_{t=0} = (aS(0) - \mu)\, I(0) < 0 \iff aS(0) < \mu \iff \frac{aS(0)}{\mu} < 1.$$

The ratio a/μ is called the basic reproduction number of the disease, and it represents the ability of spreading of this disease. To prevent a pandemic, one can reduce the ratio by lowering the transmission rate a (by quarantining infected people, washing your hands, wearing a face-mask, maintaining social distancing, etc.) or the initial fraction $S(0)$ of susceptible individuals (by implementing large-scale of vaccination). The recovered rate $\mu(t)$ is somehow hard to change since it depends mainly on the disease we are facing.

2.4 Differences Between Linear and Nonlinear Differential Equations

Keywords: fundamental results for first-order DEs, existence, uniqueness

So far we have discussed a number of initial-value problems, each of which had a solution and apparently only one solution. That raises the question of whether this is true of all initial-value problems for first-order equations. For linear equations, the answers to these questions are given by the following fundamental results.

Theorem 2.4.1 (Existence and Uniqueness for First-Order Linear Equations)**.** Consider the linear first-order differential equation with initial condition:

$$\frac{dy}{dt} + p(t)y = g(t), \quad y(t_0) = y_0. \tag{2.33}$$

If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique solution $y = \phi(t)$ that satisfies the differential equation for each t in the interval I and that also satisfies the initial condition.

Theorem 2.4.1 states that the given initial-value problem has a solution and also that the problem has only one solution. In addition, it states that the solution exists throughout any interval I containing the initial point $t = t_0$ in which the functions p and g are continuous. That is, the solution can be discontinuous or fail to exist only at points where at least one of p and g is discontinuous.

Recall in Section 2.1, we have derived that the equation (2.33) has a solution of the form

$$y(t) = \frac{1}{\mu(t)} \left(\int \mu(s)g(s)\ ds + C \right)$$

where C is a constant determined by the initial condition and

$$\mu(t) = \exp\left(\int p(t)\ dt \right).$$

Since the function p is continuous for $\alpha < t < \beta$, the function μ is defined and it is differentiable and $\mu \neq 0$.

Turning now to nonlinear differential equations, we replace Theorem 2.4.1 by a more general result, such as the one that follows.

Theorem 2.4.2 (Existence and Uniqueness for First-Order Non-Linear Equations)**.** Consider the nonlinear first-order differential equation with initial condition:

$$\frac{dy}{dt} = f(t, y) \tag{2.34}$$

If the functions f and $\partial f/\partial y$ are continuous on some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point $(t, y) = (t_0, y_0)$, then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there exists a unique solution $y = \phi(t)$ that satisfies the initial-value problem (2.34).

Observe that the hypotheses in Theorem 2.4.2 reduce to those in Theorem 2.4.1 if the differential equation is linear. In this case, we have

$$f(t,y) = -p(t)y + g(t) \quad \text{and} \quad \frac{\partial f(t,y)}{\partial y} = -p(t).$$

Hence, the continuity of f and $\partial f/\partial y$ is equivalent to the continuity of p and g. Note that the conditions stated in Theorem 2.4.2 are sufficient to guarantee the existence of a unique solution of the initial-value problem (2.34) in some interval $(t_0 - h, t_0 + h)$, but they are not necessary. That is, the conclusion remains true under slightly weaker hypotheses about the function f. In fact, the existence of a solution (but not its uniqueness) can be established on the basis of the continuity of f alone.

An important geometrical consequence of the uniqueness parts of Theorems 2.4.1 and 2.4.2 is that the graphs of two solution curves cannot intersect each other. Otherwise, there would be two solutions that satisfy the initial condition corresponding to the point of intersection, in contradiction to Theorem 2.4.1 or 2.4.2. We now consider some examples.

Example 2.4.3. Use Theorem 2.4.1 to find an interval in which the initial-value problem

$$\begin{aligned} t\frac{dy}{dt} + 2y &= 4t^2, \\ y(1) &= 2, \end{aligned} \tag{2.35}$$

has a unique solution. Do the same when the initial condition is changed to $y(-1) = 2$.

Solution. Rewriting (2.35) in the standard form, we have

$$\frac{dy}{dt} + \frac{2}{t}y = 4t,$$

and we can set $p(t) = 2/t$ and $g(t) = 4t$. Thus, for this equation, g is continuous for all t, while p is continuous only for $t < 0$ or for $t > 0$. The interval $t > 0$ contains the initial point $t = 1$; consequently, Theorem 2.4.1 guarantees that the problem (2.35) has a unique solution on the interval $0 < t < \infty$. The solution of this initial-value problem is

$$y(t) = t^2 + \frac{1}{t^2} \quad \text{for any } t > 0.$$

Now suppose that the initial condition is changed to be $y(-1) = 2$. Then, Theorem 2.4.1 asserts the existence of a unique solution for $t < 0$. As you can readily verify, the solution is again given by

$$y(t) = t^2 + \frac{1}{t^2}$$

but now on the interval $t < 0$. □

Example 2.4.4. Apply Theorem 2.4.2 to the initial-value problem

$$\begin{aligned} \frac{dy}{dt} &= \frac{3t^2 + 4t + 2}{2(y-1)}, \\ y(0) &= -1. \end{aligned} \tag{2.36}$$

Repeat this analysis when the initial condition is changed to $y(0) = 1$.

Solution. One has to apply Theorem 2.4.2 since the differential equation is nonlinear. Observe that

$$f(t,y) = \frac{3t^2+4t+2}{2(y-1)} \quad \text{and} \quad \frac{\partial f}{\partial y}(t,y) = -\frac{3t^2+4t+2}{2(y-1)^2}.$$

Thus, each of these functions is continuous everywhere except on the line $y = 1$. Consequently, a rectangle can be drawn about the initial point $(t, y) = (0, -1)$ in which both f and $\partial f/\partial y$ are continuous. Therefore, Theorem 2.4.2 guarantees that the initial-value problem (2.36) has a unique solution in some interval about $t = 0$. However, even though the rectangle can be stretched infinitely far in both the positive and the negative t directions, this does not necessarily mean that the solution exists for all t. Indeed, this problem is solved in Example 2.2.2, and the solution exists only for $t > -2$.

Now suppose we change the initial condition to $y(0) = 1$. The initial point now lies on the line $y = 1$, so no rectangle can be drawn about it with which f and $\partial f/\partial y$ are continuous. Consequently, Theorem 2.4.2 says nothing about possible solutions of this modified problem. However, if we separate the variables and integrate, as in Example 2.2.2, we find that

$$y^2 - 2y = t^3 + 2t^2 + 2t + c$$

for any constant c. Further, if $t = 0$ and $y = 1$, then we obtain that

$$c = y^2 - 2y - t^3 - 2t^2 - 2t = -1.$$

Finally, by solving for y, we obtain

$$y = 1 \pm \sqrt{t^3 + 2t^2 + 2t} = 1 \pm \sqrt{t(t^2 + 2t + 2)} = 1 \pm \sqrt{t\left[(t+1)^2 + 1\right]}. \tag{2.37}$$

The equation (2.37) provides two functions that satisfy the given differential equation in (2.36) for $t > 0$ and also satisfy the initial condition $y(0) = 1$. The fact that there are two solutions to this initial value problem reinforces the conclusion that Theorem 2.4.2 does not apply to this initial-value problem. □

According to Theorem 2.4.1, the solution of the initial-value problem

$$\frac{dy}{dt} + p(t)y = g(t), \quad y(t_0) = y_0$$

exists throughout any interval about $t = t_0$ in which the functions p and g are continuous. Thus, vertical asymptotes or other discontinuities in the solution can occur only at points of discontinuity of p or g.

On the other hand, for a nonlinear initial-value problem satisfying hypotheses of Theorem 2.4.2, the interval in which a solution exists may be difficult to determine. The solution $y = \phi(t)$ is certain to exist as long as the point $(t, \phi(t))$ remains within a region in which the hypotheses of Theorem 2.4.2 are satisfied. This is what determines the value of h in that theorem. However, since $\phi(t)$ is usually not known, it may be impossible to locate the point $(t, \phi(t))$ with respect to this region. In any case, the interval in which a solution exists may have no simple relation to the function f in the differential equation $y' = f(t, y)$. This is illustrated by the following example.

Example 2.4.5. Solve the (nonlinear) initial-value problem

$$\frac{dy}{dt} = y^2, \quad y(0) = 1, \tag{2.38}$$

and determine the interval in which the solution exists.

Solution. Theorem 2.4.2 guarantees that this problem has a unique solution since $f(t, y) = y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere. To find the solution, we separate the variables and integrate with the result that

$$\frac{dy}{y^2} = dt \implies -\frac{1}{y} = t + c.$$

Then, solving for y, we have

$$y = -\frac{1}{t+c}.$$

To satisfy the initial condition, the constant c satisfies

$$1 = y(0) = -\frac{1}{c} \implies c = -1 \implies y = \frac{1}{1-t}. \tag{2.39}$$

Clearly, the solution becomes unbounded as $t \to 1$; as a result, the solution exists only in the interval $-\infty < t < 1$. There is no indication from the differential equation itself, however that the point $t = 1$ is in any way remarkable. Moreover, if the initial condition is replaced by $y(0) = y_0$ for some given constant y_0, then the the solution of the initial-value problem is

$$y(t) = \frac{y_0}{1 - y_0 t}. \tag{2.40}$$

The solution (2.40) becomes unbounded as $t \to 1/y_0$, so the interval of existence of the solution is $-\infty < t < 1/y_0$ if $y_0 > 0$, and is $1/y_0 < t < \infty$ if $y_0 < 0$. This example illustrates another feature of initial-value problems for nonlinear equations: the singularities of the solution may depend in an essential way on the initial conditions as well as on the differential equation. □

To summarize, the linear differential equation

$$\frac{dy}{dt} + p(t)y = g(t)$$

has the following nice properties:

- Assuming that the coefficients are continuous, there is a general solution containing an arbitrary constant. A particular solution that satisfies a given initial condition can be picked out by choosing the proper value of the arbitrary constant.

- There is an expression for the solution. Also, the expression of the solution is an explicit one for the solution $y = \phi(t)$.

- The possible points of discontinuity, or singularities, of the solution can be identified (without solving the problem) merely by finding the points of discontinuity of the coefficients. Thus, if the coefficients are continuous for all t, then the solution also exists and is differentiable for all t.

None of these statements are true, in general, of nonlinear equations. Although a nonlinear equation may well have a solution involving an arbitrary constant, there may also be other solutions. There is no general formula for solutions of nonlinear equations. The expression might be implicit. Finally, the singularities of solutions of nonlinear equations can usually be found only by solving the equation and examining the solution. It is likely that the singularities will depend on the initial condition and on the differential equation.

2.5 Autonomous Differential Equations

Keywords: autonomous, equilibrium, stability, phase line, graphical analysis

In this section, we introduce and consider an important class of first-order equations of the following form:

$$\frac{dy}{dt} = f(y). \tag{2.41}$$

Such equations are called **autonomous** in which the independent variable t does not appear explicitly. For instance, the equation

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

that we considered in Chapter 1 is an autonomous equation although $v = v(t)$ is still a function of t, and t does not appear on the right-hand side of the above equation. We treat v as a single variable.

Note that the equation (2.41) is separable, so the discussion in Section 2.2 is applicable to it. The main purpose of this section is to show how geometric methods can be used to obtain important qualitative information directly form the differential equation without solving it. Of fundamental importance in this effort are the concept of **equilibrium** of the underlying differential equation and **its stability**.

We first introduce the concept of **equilibrium solution** of the autonomous differential equation (2.41) with given right-hand side $f(y)$. Suppose that the constant function $y(t) = c$ (here c is any given number) such that $f(y) = 0$, then this constant function $y(t) = c$ is called an **equilibrium solution** (or simply an equilibrium) of (2.41). Note that for equilibrium solution $y = c$, it must have $y'(t) \equiv 0$.

We now present some examples to demonstrate what kind of information can be obtained by performing some graphical analysis related to the so-called phase diagram and phase line.

Example 2.5.1. Consider the equation $y' = 9.8 - 0.2y$. It is an autonomous equation. By solving the equation (finding the value of y as a single variable), we get

$$9.8 - 0.2y = 0 \iff y = 49.$$

Hence, $y = y(t) = 49$, which is a constant function, is an equilibrium to this differential equation. In fact, once we obtain the equilibrium, we can draw a so-called **phase diagram** and the **phase line** (i.e. the line $y = 49$). It is another way to visualize the differential equation (similar to direction field, but slightly different).

Figure 2.3 demonstrates a phase diagram and the (horizontal) line $y = 49$, indicating the equilibrium $y = 49$, is a phase line. Graphically speaking, the phase line splits the yt-plane into two regions: the region where $y > 49$ and the one $y < 49$. We take a closer talk at these two cases:

1. We examine the case where $y > 49$ (upper region above the phase line). For instance, we take an initial condition, say $y(0) = 60$ and accompany it with the differential equation. The solution to this IVP is $y(t) = 49+11e^{-t/5}$ and the red curve in Figure 2.3 is the solution curve to this IVP. Note that the derivative y' can be computed by the DE $y' = 9.8 - 0.2y$. In this region, since y is always bigger than 49, it implies that the derivative is always negative (i.e. $y' < 0$). That means, the function value of y is decreasing and it approaches to the equilibrium $y = 49$ as $t \to \infty$ (as shown in Figure 2.3). We draw an arrow pointing downward in this region to indicate y is decreasing in this region.

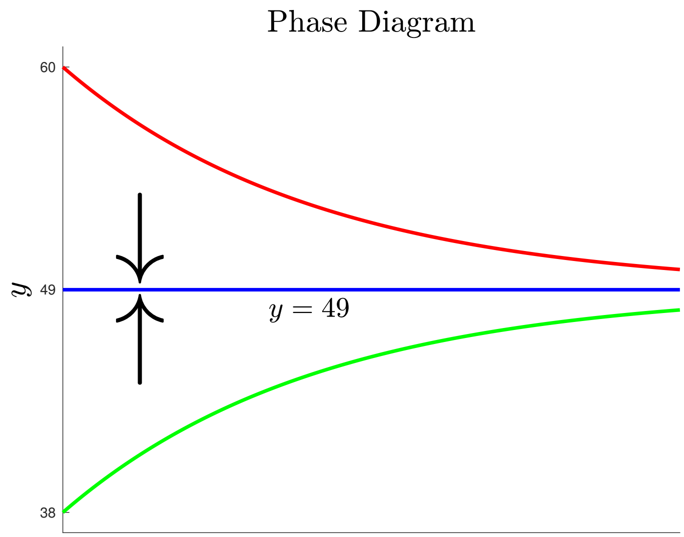


Figure 2.3: The phase diagram and phase line $y = 49$ in Example 2.5.1; red curve: $y(t) = 49 + 11e^{-t/5}$ with $y(0) = 60$; blue curve: $y(t) = 49 - 11e^{-t/5}$ with $y(0) = 38$.

2. The another case where $y < 49$ (lower region below the phase line) can be analyzed in a similar manner. For instance, we take an initial condition $y(0) = 38$ and accompany it with the differential equation. The solution to this IVP is $y(t) = 49 - 11e^{-t/5}$ and the green curve in Figure 2.3 is the corresponding solution curve. Since $y < 49$, it implies that $y' > 0$ and the function values of y is increasing and it approaches to the equilibrium $y = 49$ as $t \to \infty$. We draw an arrow pointing upward in this region to indicate y is increasing in this region.

To summarize, with only the DE $y' = 9.8 - 0.2y$, we can find its equilibrium solution $y = 49$ and this equilibrium $y = 49$ can be viewed as the solution to the IVP with initial condition $y(0) = 49$. When we consider the same DE but with different initial conditions (say $y(0) = 60$ or $y(0) = 38$ in the previous cases), the solutions to the IVPs for both cases approach the equilibrium $y = 49$ as $t \to \infty$. Graphically speaking, one arrow pointing downward and one upward (like squeezing sandwich, where those arrows are bread, and the phase line is ingredient) can be observed. In this case, we say that the equilibrium $y = 49$ is **stable**.

We can further interpret the meaning of being stable for an equilibrium. When the equilibrium is stable, if we pick an initial condition that is close to the equilibrium (but not exactly), then the solution starting from that picked initial condition approaches to this stable equilibrium as $t \to \infty$.

Example 2.5.2. Consider the differential equations $y' = (1 - y)(3 - y)$. Solving the equation $(1 - y)(3 - y) = 0$ gets $y = 1$ or $y = 3$. Hence, this equation has two equilibriums. Accordingly, we can draw two phase lines $y = 1$ and $y = 3$ on the phase diagram. These two lines split the plane into three different regions: $y < 1$; $1 < y < 3$; and $y > 3$.

1. When $y < 1$: the derivative $y' = (1 - y)(3 - y) > 0$; thus the value of y is increasing. One can draw an arrow pointing upward to the phase line $y = 1$.

2. When $1 < y < 3$: the derivative $y' = (1 - y)(3 - y) < 0$ (as $1 - y < 0$ and $3 - y > 0$, their product is negative); thus the value of y is always decreasing. One draws an arrow pointing down to the phase line $y = 1$ and draws another arrow pointing down near another phase line $y = 3$.

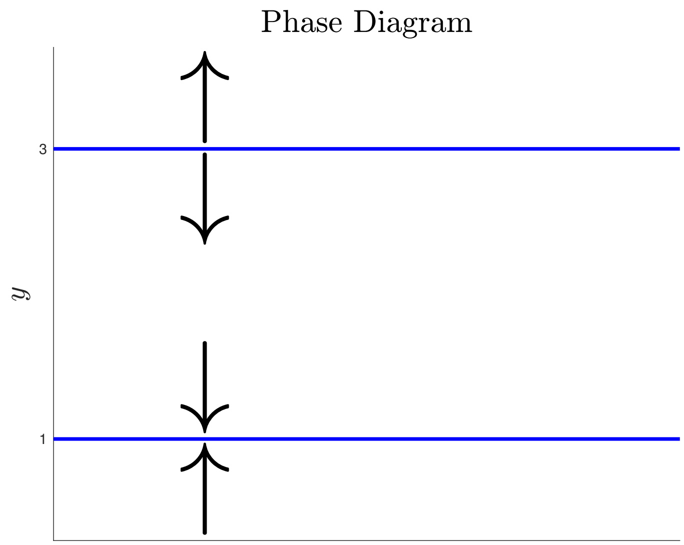


Figure 2.4: The phase diagram and phase lines $y = 1$ and $y = 3$ in Example 2.5.2. The equilibrium $y = 1$ is stable (like squeezing sandwich) while $y = 3$ is unstable (like bread goes away).

3. Similarly, when $y > 3$, the derivative is positive; thus the value of y is increasing and we draws an arrow pointing upward near above the phase line $y = 3$.

From the phase diagram Figure 2.4, we can see that the equilibrium $y = 1$ is stable. That means if we pick an initial condition that is close to the value of 1 (but not exceed 3 in this example), then our solution starting from this initial condition always approaches to 1 as $t \to \infty$. On the other hand, $y = 3$ is called **unstable** since no matter where we start our solution around the value of $y = 3$, the solution with this starting point always goes away from $y = 3$. Graphically speaking, one arrow on top pointing upward but another arrow below pointing downward (like bread goes away) can be observed in the unstable case.

To summarize, we classify the equilibrium solutions as follows. Suppose that $y(t) = c$ is an equilibrium solution of the autonomous differential equation (2.41). Then, we call $y(t) = c$ is:

- **Stable:** if all solutions of (2.41) with initial conditions y_0, which is near $y = c$, approach c as $t \to \infty$ (like squeezing sandwich).
- **Unstable:** if all solutions with initial conditions y_0, which is near $y = c$, do not approach c as $t \to \infty$ (bread goes away).
- **Semi-stable:** if solutions $y(t)$ with initial conditions y_0 on one side of c approach c as $t \to \infty$, while solutions with initial conditions y_0 on the other side of c do not approach c as $t \to \infty$ (the case between stable and unstable).

Remark: Using the fundamental results of first-order differential equations in 2.4, if $f(y)$ and $\partial f/\partial y$ are continuous, then equation (2.41) has one and only one solution. That means any solution curve of the differential equation (2.41) cannot intersect the one that of the equilibrium solution.

We summarize a step-by-step procedure for performing graphical analysis in terms of equilibriums of DEs. Given an autonomous equation

$$\frac{dy}{dt} = f(y)$$

with $f(y)$ and $\partial f/\partial y$ being continuous, we follow the steps below to classify equilibrium solutions:

1. Solve $f(y) = 0$ to find out all equilibrium solution(s).
2. For each equilibrium, study the values of $f(y)$ around the equilibrium solution(s). Draw a line (**phase line**) that represents the values of y and make tick marks at equilibrium values.
3. Between tick marks determine if $f(y)$ is positive or negative.
   - If $f(y)$ is positive, then dy/dt is positive so any solution y in this region is increasing.
   - If $f(y)$ is negative, then dy/dt is negative so any solution y in this region is decreasing.
4. Classify the equilibriums:
   - If increasing below and decreasing above $\Longrightarrow$ stable.
   - If decreasing below and increasing above $\Longrightarrow$ unstable.
   - If decreasing below and above **OR** increasing below and above $\Longrightarrow$ semi-stable.

Example 2.5.3. Find and classify the equilibrium point(s) of

$$\frac{dy}{dt} = -(y-10)^2(y-4).$$

Solution. The equilibrium points are $y = 4$ and $y = 10$.

- For $y > 10$, dy/dt is negative, so $y(t)$ is decreasing.
- For $4 < y < 10$, dy/dt is negative, so $y(t)$ is decreasing.
- For $y < 4$, dy/dt is positive, so $y(t)$ is increasing.

As a result, $y(t) = 4$ is stable and $y(t) = 10$ is semi-stable. □

Example 2.5.4. Find and classify the equilibrium point(s) of

$$\frac{dy}{dt} = (y^3 - 8)(e^y - 1).$$

Solution. The equilibrium points are $y = 2$ and $y = 0$.

- For $y > 2$, dy/dt is positive, so $y(t)$ is increasing.
- For $0 < y < 2$, dy/dt is negative, so $y(t)$ is decreasing.
- For $y < 0$, dy/dt is positive, so $y(t)$ is increasing.

As a result, $y(t) = 0$ is stable and $y(t) = 2$ is unstable. □

Example 2.5.5 (Exponential growth)**.** We consider the problem of population growth of a given species at specific time interval. Let $y = \phi(t)$ be the population of the given species at time t. The simplest hypothesis concerning the variation of population is that the rate of change of y is proportional to the current value of y; that is,

$$\frac{dy}{dt} = ry, \tag{2.42}$$

where the constant r is called the **rate of growth** or **decline**, depending on the sign of r. Here, we assume that the population is growing and thus $r > 0$. Solving (2.42) subject to the initial condition $y(0) = y_0$ we obtain

$$y(t) = y_0 e^{rt}. \tag{2.43}$$

Thus, the mathematical model consisting of the initial-value problem with $r > 0$ predicts that the population will grow exponentially for all times. Under ideal conditions, the solution (2.43) has been observed to be reasonably accurate for many populations, at least for limited periods of time. However, it is clear that such ideal conditions cannot continue indefinitely; eventually, limitations on space, food supply, or other resources will reduce the growth rate and bring an end to uninhibited exponential growth.

Example 2.5.6 (logistic growth)**.** To take account of the fact that the growth rate actually depends on the population, we consider

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y. \tag{2.44}$$

The equation (2.44) is known as the **logistic equation**. The constant r is called the **intrinsic growth rate**.

We now perform the so-called **graphical analysis** to give a sketch of the solution to (2.44). The same methods also apply to the more general equation (2.41).

We first seek equilibrium solutions of (2.44). Thus, the equilibrium solutions are $y = 0$ and $y = K$. The graph of $f(y)$ is a parabola shown in Figure 2.5. The intercepts are $(0, 0)$ and $(K, 0)$, corresponding to the equilibrium solutions of (2.44), and the vertex of the parabola is $(K/2, rK/4)$.

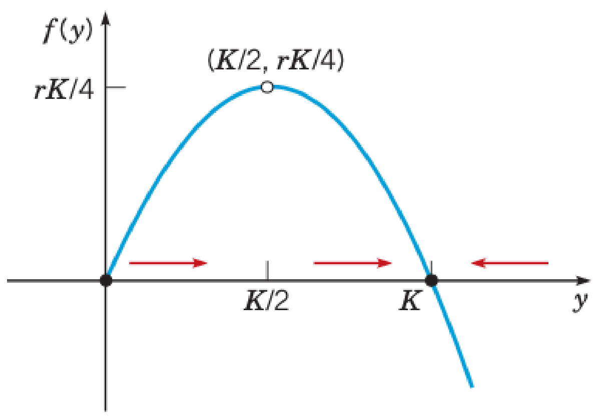


Figure 2.5: The function $f(y) = r(1 - y/K)y$ versus y for logistic model.

For the logistic equation (2.44), one can observe that

- The equilibrium solutions of (2.44) are $y = 0$ and $y = K$.
- The function $f(y) = r(1 - y/K)y$ achieves maximum $f(y^*) = rK/4$ when $y^* = K/2$.
- If $0 < y < K$, then $dy/dt > 0$; thus the solution is increasing; if $y > K$, then $dy/dt < 0$; thus the solution is decreasing.
- Hence, $y = 0$ is *unstable* and $y = K$ is *stable*.

The dots at $y = 0$ and $y = K$ are the critical points, or equilibrium solutions. The arrows again indicate that y is increasing whenever $0 < y < K$ and that y is decreasing whenever $y > K$. Further from Figure 2.5, note that if y is near zero or K, then the slope $f(y)$ is near zero, so the solution curves are relatively flat. They become steeper as the value of y leaves the neighborhood of zero or K.

To carry out further investigation of the solution of (2.44), one can determine the concavity of the solution curves and the location of inflection points by finding the second order derivative d^2y/dt^2. To do this, we apply the chain rule:

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt}f(y) = f'(y)\frac{dy}{dt} = f'(y)f(y). \tag{2.45}$$

The signs of f and f' can be easily identified from the graph of $f(y)$ versus y. We note that

- The solution curves is concave up when $y'' > 0$; that is, f and f' has the same sign.
- The solution curves is concave down when $y'' < 0$; that is, f and f' has the opposite signs.
- Inflection point(s) may occur when $f'(y) = 0$.

If we consider the logistic model (2.44) and we recall $f(y) = r(1 - y/K)y$, then we have

$$f'(y) = r\left(1 - \frac{2y}{K}\right).$$

The inflection point is $y = K/2$. Hence, solutions are concave up when $0 < y < K/2$ where f is positive and increasing, so that both f and f' are positive. Solutions are also concave up for $y > K$ where f is negative and decreasing (both f and f' are negative). For $K/2 < y < K$, solutions are concave down since here f is positive and decreasing, so f is positive but f' is negative.

We recall that $y = 0$ and $y = K$ are two equilibrium solutions of the logistic model (2.44). The uniqueness part of Theorem 2.4.2 states that only one solution can pass through a givne point in the ty-plane. Thus, other solutions may be asymptotic to the equilibrium solution(s) as $t \to \infty$, they cannot intersect it at any finite time. Consequently, a solution that starts in the interval $0 < y < K$ remains in this interval for all time, and similarly for a solution that starts in $K < y < \infty$.

If a solution $y = y(t)$ starts in the interval $y > K$, then K is the lower bound of the solution of the logistic model. While if a solution starts in the interval $0 < y < K$, then K is the upper bound that is approached but not exceeded. Thus it is natural to refer K as the **saturation level** or the **environmental carrying capacity** for the given species.

2.6 Exact Differential Equations and Integrating Factors

Keywords: exact differential equations, implicit function

We first look at an example below.

Example 2.6.1. Solve the differential equation

$$2x + y^2 + 2xy\frac{dy}{dx} = 0. \tag{2.46}$$

Solution. The equation is neither linear nor separable, so the methods suitable for those types of equations are not applicable here. However, observe that the function $\psi(x, y) = x^2 + xy^2$ has the property that

$$\frac{\partial \psi}{\partial x} = 2x + y^2, \quad \frac{\partial \psi}{\partial y} = 2xy.$$

Therefore, the differential equation (2.46) becomes

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0.$$

Assuming that y is a function of x, we use the chain rule to write the left-hand side of the equation above as $d\psi(x, y)/dx$. Then, the equation becomes

$$\frac{d\psi}{dx}(x, y) = \frac{d}{dx}(x^2 + xy^2) = 0.$$

Integrating with respect to x, we obtain

$$x^2 + xy^2 = C \tag{2.47}$$

for any arbitrary constant C. The level curves of $x^2 + xy^2 = C$ for different values of C are the integral curves of equation (2.46) and solutions of (2.46) are defined implicitly by (2.47). □

In general, we consider the following differential equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{2.48}$$

Suppose that we can identify a function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x}(x, y) = M(x, y), \quad \frac{\partial \psi}{\partial y}(x, y) = N(x, y),$$

and such that $\psi(x, y) = c$ defines $y = \phi(x)$ implicitly as a differentiable function of x. When there is a function $\psi(x, y)$ such that $\psi_x = M$ and $\psi_y = N$, we can write

$$M(x, y) + N(x, y)\frac{dy}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}(\psi(x, y))$$

and the differential equation (2.48) becomes

$$\frac{d}{dx}(\psi(x, y)) = 0. \tag{2.49}$$

In this case (2.49) is called an **exact differential equation** because it can be expressed exactly as the derivative of a specific function. Solutions of (2.48) or the equivalent form (2.49) are given implicitly by

$$\psi(x, y) = C$$

where C is an arbitrary constant.

Remark: A differential equation (2.48) is exact if and only if the functions $M(x, y)$ and $N(x, y)$ satisfies

$$M_y(x, y) = N_x(x, y).$$

In this case, there exists a function ψ such that $\psi_x = M$ and $\psi_y = N$.

Example 2.6.2. Solve the following differential equation

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)\frac{dy}{dx} = 0 \tag{2.50}$$

Solution. This is an exercise we have seen in multi-variable calculus. We set $M(x,y) = (y\cos x + 2xe^y)$ and $N(x,y) = \sin x + x^2e^y - 1$. First, we calculate M_y and N_x:

$$M_y(x,y) = \cos x + 2xe^y, \quad N_x = \cos x + 2xe^y.$$

The equation (2.50) is exact since $M_y = N_x$. Thus, there is a function $\psi(x,y)$ such that

$$\psi_x = y\cos x + 2xe^y, \tag{2.51}$$
$$\psi_y = \sin x + x^2e^y - 1. \tag{2.52}$$

Integrating (2.51) with respect to x, we obtain

$$\psi(x,y) = y\sin x + x^2e^y + h(y)$$

for some function $h(y)$ depending only on y. Next, from the formula above, we calculate ψ_y and obtain

$$\psi_y(x,y)\sin x + x^2 + e^y + h'(y) = \sin x + x^2e^y - 1.$$

Therefore, we obtain that $h'(y) = -1$ and $h(y) = -y$. The constant of integration here can be omitted since any solution of the preceding differential equation is satisfactory; we do not require the most general one. Substituting for h in the formula of ψ, we obtain

$$\psi(x,y) = y\sin x + x^2e^y - y.$$

Hence, the general solution of (2.50) is given by

$$y\sin x + x^2e^y - y = C$$

for any arbitrary constant C. □

Sometimes it is possible to convert a non-exact differential equation into an exact one by multiplying the equation by a suitable integrating factor. For instance, if the equation

$$M(x,y) + N(x,y)y' = 0$$

is not exact, then we multiply the above equation by the so-called *integrating factor* $\mu(x,y)$ such that it becomes exact. That is,

$$\mu M + \mu N y' = 0$$

is exact and μ satisfies

$$(\mu M)_y = (\mu N)_x \iff M\mu_y - N\mu_x + (M_y - N_x)\mu = 0.$$

If the integrating factor μ is only a function of x, that is $\mu_y = 0$, then we have

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu.$$

If $(M_y - N_x)/N$ is a function of x only, then there is an integrating factor μ that also depends only on x.

Example 2.6.3. Solve the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0 \tag{2.53}$$

Solution. The equation (2.53) is not exact since

$$(3xy + y^2)_y = 3x + 2y \neq (x^2 + xy)_x = 2x + y.$$

However, we can find an integrating factor $\mu(x)$ for this problem. Since

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x},$$

there is an integrating factor μ that is a function of x only and μ satisfies

$$\mu'(x) = \frac{\mu}{x} \implies \mu(x) = x.$$

Multiplying $\mu(x) = x$ to the equation (2.53), it becomes

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0.$$

This equation is exact since

$$(3x^2y + xy^2)_y = 3x^2 + 2xy = (x^3 + x^2y)_x.$$

It remains to find the function $\psi(x, y)$ such that

$$\psi_x(x,y) = 3x^2y + xy^2, \quad \psi_y(x,y) = x^3 + x^2y.$$

Thus, we have

$$\psi(x,y) = x^3y + \frac{1}{2}x^2y^2$$

and the solution of (2.53) is implicitly given by

$$x^3y + \frac{1}{2}x^2y^2 = C$$

for any arbitrary constant C. □

2.7 Exercises

There are 4 questions in this assignment. Answer all. Please write down your name and UIN. The deadline is **5:00 pm (CDT), Sep 16 2022**.

1. Consider the following first-order linear differential equation

$$\frac{dy}{dt} + \frac{t}{2}y = 1. \tag{2.54}$$

   (a) Find the general solution of (2.54) using method of integrating factor.

   (b) Based on the result of (a), show that the general solution of (2.54) tends to a limit as $t \to \infty$ and find the limiting value. **Hint:** use L'Ĥopital's rule.

2. Find the solution of the given initial value problem in explicit form

$$\frac{dy}{dx} = (1-2x)y^2, \quad y(0) = -\frac{1}{6}. \tag{2.55}$$

3. Consider the following initial-value problem:

$$\begin{aligned} \frac{dy}{dt} &= \frac{-t + \sqrt{t^2 + 4y}}{2}, \\ y(2) &= -1. \end{aligned} \tag{2.56}$$

   (a) Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of (2.56). Find the intervals where these solutions valid.

   (b) Briefly explain why the existence of two solutions of the given problem above does not contradict the fundamental results that we explained in class.

4. Consider the initial value problem

$$\frac{dy}{dt} = y^{1/3} \quad y(0) = 0. \tag{2.57}$$

   for $t \geqslant 0$. Find the solution(s) to the initial-value problem. Can you find more than one solution?

Exercises

There are 5 questions in this assignment. Answer all. Please write down your name and UIN. The deadline is **11:59 pm (CDT), Sep 30 2022**.

1. Consider the following initial-value problem

$$\frac{dy}{dt} = f(y) \quad y(0) = y_0, \tag{2.58}$$

where $f(y) = \alpha y(1-y)$, and the constants $\alpha > 0$ and $y_0 > 0$ are given.

(a) (Roughly) Sketch the graph of $f(y)$ and find the equilibrium point(s) for the differential equation (2.58).

(b) Based on the information found in part (a), sketch (by hand, roughly) the graph of the solution y.

(c) Solve the initial-value problem (2.58). What is the limit of the solution $y(t)$ as $t \to \infty$

2. Show that the following problem is not exact but becomes exact when multiplied by the given integrating factor $\mu(t, y)$. Then, solve the equation.

$$t^2y^3 + t(1+y^2)\frac{dy}{dt} = 0, \quad \mu(t,y) = \frac{1}{ty^3}.$$

3. Solve the following given initial-value problems:

- $y'' + y' - 2y = 0$, $y(0) = 1$, $y'(0) = 1$.
- $4y'' - y = 0$, $y(-2) = 1$, $y'(-2) = -1$.
- $3y'' - y' + 2y = 0$, $y(0) = 2$, $y'(0) = 0$.
- $y'' - 2y' + 5y = 0$, $y(\pi/2) = 0$, $y'(\pi/2) = 2$.
- $4y'' + 4y' + y = 0$, $y(0) = 1$, $y'(0) = 2$.
- $9y'' - 12y' + 4y = 0$, $y(0) = 2$, $y'(0) = -1$.

4. Find the determinant of the Wronskian matrix $W[y_1, y_2; t]$ of the given pair of functions: $y_1(t) = \cos^2 t$, $y_2(t) = 1 + \cos(2t)$.

5. Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the differential equation

$$t^2y'' - 2y = 0 \quad t > 0.$$

Compute also the determinant of Wronskian matrix of y_1 and y_2 at any point t.

Reference Solutions of Exercises in Chapter 2

1. The general solution is

$$y(t) = e^{-t^2/4}\int_{t_0}^{t} e^{s^2/4}ds + ce^{-t^2/4}$$

for any arbitrary constants t_0 and c. Using L'Ĥopital's rule, we have

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{\int_{t_0}^{t} e^{s^2/4}\, ds}{e^{t^2/4}} = \lim_{t\to\infty} \frac{e^{t^2/4}}{e^{t^2/4}0.5t} = \lim_{t\to\infty} \frac{2}{t} = 0.$$

2. Separating variables, we have

$$\frac{dy}{y^2} = (1-2x)dx \implies -y^{-1} = x - x^2 + c \implies y(x) = \frac{1}{x^2 - x + c}.$$

Using the initial condition, we have $c = 6$.

3. (a) Direct verification.

(b) Let $f(t,y) = \dfrac{-t+\sqrt{t^2+4y}}{2}$. Then, the partial derivative $\dfrac{\partial f}{\partial y} = \dfrac{2}{\sqrt{t^2+4y}}$ is not continuous at $(t_0, y_0) = (2,-1)$. As a result, the function $f(t,y)$ does not satisfy the conditions in Theorem 2.4.2 and does not contradict the uniqueness part of Theorem 2.4.2.

4. Separating variables, we have

$$\frac{dy}{y^{1/3}} = dt \implies \frac{3}{2}y^{2/3} = t + c \implies y^{2/3} = \frac{2}{3}t + c \implies y^2 = \frac{8}{27}t^3 + \frac{4c}{3}t^2 + 2c^2t + c^3.$$

Using the initial condition, we have $c = 0$. Then, the solution y satisfies $y^2 = \frac{8}{27}t^3$ and we have two solutions

$$y_1(t) = \frac{2\sqrt{6}}{9}t^{3/2} \quad \text{and} \quad y_2(t) = -\frac{2\sqrt{6}}{9}t^{3/2}.$$

5. Solving the equation, we obtain

$$\frac{dy}{y(1-y)} = \alpha dt \implies \left(\frac{1}{y} - \frac{1}{y-1}\right)dy = \alpha dt \implies \log\left(\frac{y}{y-1}\right) = \alpha t + c,$$

$$1 + \frac{1}{y-1} = Ce^{\alpha t} \implies y(t) = 1 + \frac{1}{Ce^{\alpha t} - 1}.$$

Using the initial condition, we have $C = \dfrac{y_0}{y_0 - 1}$. The limit of the solution is

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(1 + \frac{y_0 - 1}{y_0e^{\alpha t} - y_0 + 1}\right) = 1.$$

6. Direction verification. The general solution is

$$\frac{t^2}{2} - \frac{1}{2y^2} + \log(y) = C$$

for any constant C.

2.8 Suggested Practice Problems

These are suggested practice problems; no need to hand in.

1. Solve the following initial-value problem:

$$\frac{dy}{dt} + \frac{1}{4}y = 3 + 2\cos(2t), \quad y(0) = 0.$$

2. Let a and λ be positive constants, and b be any real number. Solve the differential equation:

$$\frac{dy}{dt} + ay = be^{-\lambda t}.$$

Discuss the cases when $a \neq \lambda$ and $a = \lambda$ separately.

3. Solve the differential equation:

$$\frac{dy}{dt} + y^2 \sin(t) = 0.$$

4. Solve the differential equation:

$$\frac{dy}{dt} = \cos^2(x)\cos^2(2y).$$

5. Solve the initial-value problem:

$$\frac{dy}{dt} = 2y^2 + xy^2 \quad y(0) = 1.$$

6. Determine an interval in which the solution of the given initial-value problem is certain to exist.

$$(t-3)\frac{dy}{dt} + (\log t)y = 2t \quad y(1) = 2.$$

7. Solve the following initial-value problem and determine how the interval in which the solution exists depends on the initial-value y_0:

$$\frac{dy}{dt} = \frac{t^2}{y(1+t^3)} \quad y(0) = y_0.$$

8. The following equation is of them form $y' = f(y)$. Sketch the graph of $f(y)$ versus y, determine the equilibrium points, and classify each one as asymptotically stable, semistable, or unstable. Draw the phase line, and sketch several graphs of solutions in the ty-plane.

$$\frac{dy}{dt} = y(1-y^2).$$

Also, find the general solution to verify whether your sketch of solutions in the ty-plane is correct or not.

9. Find the value of b for which the given equation is exact, and then solve it using that value of b.

$$(ty^2 + bt^2y) + (t+y)t^2\frac{dy}{dt} = 0.$$

Chapter 3

Second-Order Linear Differential Equations

3.1 Homogeneous Equations with Constant Coefficients

Keywords: homogeneous equation, forcing function, characteristic equation

We study the following second-order linear differential equation:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \tag{3.1}$$

where $p(t)$, $q(t)$, and $g(t)$ are given and only depend on t but not y. Sometimes we write y' and y'' to represent the first- and second-order derivatives of y with respect to t. In discussing (3.1), and in trying to solve it, we restrict to intervals where $p(t)$, $q(t)$, and $g(t)$ are continuous. We only cover little nonlinear second-order differential equation because of its difficulty to solve analytically. The equation (3.1) is usually equipped with an initial condition of the following form:

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{3.2}$$

where t_0, y_0, and y_0' are given numbers. A second-order differential equation needs two initial conditions because it involves two integrations to find a solution and each integration introduces an arbitrary constant. The equation (3.1) is called

- **homogeneous** if $g(t) \equiv 0$ for all t.
- Otherwise, it is called **nonhomogeneous**. In this case, the right-hand side function $g(t)$ is usually called the **forcing function** since in many applications it describes an externally applied force.

In this section, we concentrate on equation with the form when $p(t)$ and $q(t)$ are constants, and $g(t) \equiv 0$. That is, we will mainly consider the following form of second-order homogeneous equation with initial conditions:

$$ay'' + by' + cy = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{3.3}$$

where $a \neq 0$, b, and c are given constants. Here, we state the steps of solving (3.3).

1. Suppose that the solution of (3.3) is of the form $y(t) = e^{rt}$ for some constant r to be determined.

2. Plug $y(t) = e^{rt}$ into the equation (3.3). It follows that $y'' = r^2 e^{rt}$ and $y' = re^{rt}$. The equation (3.3) becomes
$$(ar^2 + br + c)e^{rt} = 0.$$
Since $e^{rt} \neq 0$, it is equivalent to
$$ar^2 + br + c = 0. \tag{3.4}$$
This is called the **characteristic equation** for (3.3). The characteristic equation has two roots (which may be real and different, complex conjugates, or real but repeated).

3. Solve (3.4) and find out its roots (denoted as r_1 and r_2).

4. Assume that $r_1 \neq r_2$. The general solution of (3.1) is
$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{3.5}$$
where c_1 and c_2 are arbitrary constants.

5. To determine the values of c_1 and c_2, we have to make use of the initial conditions
$$y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y_0'.$$
We set $t = t_0$ and $y = y_0$ in (3.5) and obtain
$$c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0. \tag{3.6}$$
Moreover, we calculate $y'(t)$ using the formula of general solution and set $t = t_0$ and $y'(t_0) = y_0'$. We obtain
$$y'(t) = r_1 c_1 e^{r_1 t} + r_2 c_2 e^{r_2 t}$$
and
$$r_1 c_1 e^{r_1 t_0} + r_2 c_2 e^{r_2 t_0} = y_0'. \tag{3.7}$$

6. Solving (3.6) and (3.7), we obtain the values of c_1 and c_2:
$$c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0} \quad \text{and} \quad c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}.$$

Remark: we consider the cases when r_1 and r_2 are complex conjugates and $r_1 = r_2$ in Sections 3.3 and 3.4.

Example 3.1.1. Solve the equation
$$y'' - y = 0. \tag{3.8}$$
Also find the solution that satisfies the initial conditions $y(0) = 2$ and $y'(0) = -1$.

Solution. First, we write the characteristic equation of (3.8). Let $y(t) = e^{rt}$ be the solution. Then, we substitute it in (3.8) and obtain

$$(r^2 - 1)e^{rt} = 0 \iff r^2 - 1 = 0.$$

Then, we obtain two roots of the characteristic equation: $r_1 = 1$ and $r_2 = -1$. Hence, the general solution of (3.8) is

$$y(t) = c_1e^t + c_2e^{-t} \implies y'(t)c_1e^t - c_2e^{-t}.$$

Here, c_1 and c_2 are arbitrary constants. To find the values of c_1 and c_2, we make use of the initial conditions $y(0) = 2$ and $y'(0) = -1$. Thus, we have

$$2 = c_1 + c_2 \quad \text{and} \quad -1 = c_1 - c_2.$$

Solving c_1 and c_2, we obtain

$$c_1 = \frac{1}{2} \quad \text{and} \quad c_2 = \frac{3}{2}.$$

The solution of the initial-value problem is

$$y(t) = \frac{1}{2}e^t + \frac{3}{2}e^{-t}.$$

□

Example 3.1.2. Find the general solution of

$$y'' + 5y' + 6y = 0. \tag{3.9}$$

Assume that the solution satisfies the initial conditions $y(0) = 2$ and $y'(0) = 3$. Find the solution of the initial-value problem.

Solution. First, we write the characteristic equation of (3.9). It becomes

$$r^2 + 5r + 6 = (r + 2)(r + 3) = 0.$$

Thus, the roots of the characteristic equation are $r = -2$ and $r = -3$. Therefore, the general solution of (3.9) is

$$y(t) = c_1e^{-2t} + c_2e^{-3t} \implies y'(t) = -2c_1e^{-2t} - 3c_2e^{-3t}.$$

where c_1 and c_2 are arbitrary constants. To determine the values of c_1 and c_2, we make use of the initial conditions. Thus, we obtain

$$2 = c_1 + c_2 \quad \text{and} \quad 3 = -2c_1 - 3c_2.$$

Solve the equations above, we obtain $c_1 = 9$ and $c_2 = -7$. Therefore, the solution of initial-value problem is

$$y(t) = 9e^{-2t} - 7e^{-3t}.$$

□

Example 3.1.3. Find the solution of the initial-value problem:

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}. \tag{3.10}$$

Solution. First, we write the characteristic equation of (3.10). It becomes

$$4r^2 - 8r + 3 = (2r-3)(2r-1) = 0.$$

Thus, we have $r_1 = 3/2$ and $r_2 = 1/2$. The general solution of (3.10) is

$$y(t) = c_1 e^{3t/2} + c_2 e^{t/2} \implies y'(t) = \frac{3}{2} c_1 e^{3t/2} + \frac{1}{2} c_2 e^{t/2}.$$

Applying the initial conditions, we obtain

$$c_1 + c_2 = 2 \quad \text{and} \quad \frac{3}{2} c_1 + \frac{1}{2} c_2 = \frac{1}{2}.$$

Then, we obtain that $c_1 = -1/2$ and $c_2 = 5/2$. The solution of the initial-value problem is

$$y(t) = -\frac{1}{2} e^{3t/2} + \frac{5}{2} e^{t/2}.$$

□

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

Keywords: Wronskian, Fundamental set of Solutions, Superposition

We state the fundamental results of second-order linear differential equation with initial conditions.

Theorem 3.2.1. The initial-value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{3.11}$$

where $p(t)$, $q(t)$, and $g(t)$ are continuous on an open interval I that contains t_0, has exactly one solution $y = \phi(t)$, and the solution exists throughout the interval I.

Example 3.2.2. Consider the initial-value problem:

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0, \quad y(1) = 2, \quad y'(1) = 1. \tag{3.12}$$

Analyze the existence and uniqueness of the initial-value problem using Theorem 3.2.1.

Solution. We first divide the equation by $t^2 - 3t = t(t-3)$ and obtain

$$y'' + \frac{1}{t-3} y' - \frac{t+3}{t(t-3)} y = 0.$$

We have $p(t) = 1/(t-3)$ and $q(t) = -(t+3)/[t(t-3)]$. The only points of discontinuity of the functions are $t = 0$ and $t = 3$. The longest open interval, containing the initial point $t = 1$, in which all coefficients are continuous is $0 < t < 3$. This is the longest interval in which Theorem 3.2.1 guarantees that the solution exists. □

To simply notation, we denote

$$L[y] := y'' + py' + qy$$

and the value of $L[y]$ at a point t is denoted as

$$L[y](t) := y''(t) + p(t)y'(t) + q(t)y(t).$$

Assume that y_1 and y_2 are two solutions of the differential equation:

$$L[y] = y'' + py' + qy = 0. \tag{3.13}$$

It is easy to observe that any linear combination of y_1 and y_2 is the solution of the differential equation (3.13). We have the following result.

Theorem 3.2.3. Let y_1 and y_2 be two solutions of the differential equation (3.13). Then, any linear combination $c_1y_1 + c_2y_2$ of y_1 and y_2 is also a solution of the differential equation (3.13), for any values of c_1 and c_2.

Theorem 3.2.3 states that, starting with only two solutions (y_1 and y_2) of (3.13), we can construct an infinite family of solutions by means of its linear combination

$$y(t) = c_1y_1(t) + c_2y_2(t).$$

Now to determine the values of c_1 and c_2, we apply the initial condition

$$y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

These initial conditions require c_1 and c_2 to satisfy

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= y_0, \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= y_0'. \end{aligned} \tag{3.14}$$

Define the **Wronskian determinant** (or **Wronskian**) of y_1 and y_2 at t_0:

$$W = W[y_1, y_2; t_0] := \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0). \tag{3.15}$$

- If $W \neq 0$, then (3.14) has a unique solution (c_1, c_2) regardless of the values of y_0 and y_0'. The values of c_1 and c_2 are given by

$$\begin{aligned} c_1 &= \frac{y_0y_2'(t_0) - y_0'y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)} = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W}, \\ c_2 &= \frac{-y_0y_1'(t_0) + y_0'y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)} = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W}. \end{aligned} \tag{3.16}$$

  With these values of c_1 and c_2, the linear combination $y = c_1y_1(t) + c_2y_2(t)$ satisfies the differential equation and the initial conditions.

- If $W = 0$, the equation (3.14) has no solution unless y_0 and y_0' have values that also make the numerators in (3.16) equal to zero. Thus, if $W = 0$, there are many initial conditions that cannot be satisfied no matter how c_1 and c_2 are chosen.

We have the following result to summarize the existence and uniqueness of second-order differential equation.

Theorem 3.2.4. Let y_1 and y_2 be two solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

and that initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

are assigned. Then, it is always possible to choose the constants c_1 and c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and the initial conditions given above if and only if the Wronskian

$$W[y_1, y_2; t_0] = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0.$$

The two-parameter family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of the differential equation if and only if there is a point t_0 where the Wronskian of y_1 and y_2 is not zero. In this case, we call y_1 and y_2 form a **fundamental set of solutions** of the differential equation if their Wronskian is nonzero.

Example 3.2.5. The differential equation

$$y'' + 5y' + 6y = 0$$

has two solutions $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$. Find the Wronskian of y_1 and y_2 for any given values of y_0.

Solution. The Wronskian of these two functions is

$$W[e^{-2t}, e^{-3t}; t_0] = \begin{vmatrix} e^{-2t_0} & e^{-3t_0} \\ -2e^{-2t_0} & -3e^{-3t_0} \end{vmatrix} = -e^{-5t_0} \neq 0.$$

Since W is nonzero for all values of t_0, the functions y_1 and y_2 can be used to construct solutions of the given differential equation, together with initial conditions prescribed at any values of t_0. In this case, $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions of the differential equation $y'' + 5y' + 6y = 0$. □

Example 3.2.6. In general, if $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

and if $r_1 \neq r_2$, then y_1 and y_2 form a fundamental set of solutions since

$$W[y_1, y_2; t_0] = \begin{vmatrix} e^{r_1 t_0} & e^{r_2 t_0} \\ r_1 e^{r_1 t_0} & r_2 e^{r_2 t_0} \end{vmatrix} = (r_2 - r_1)e^{(r_1+r_2)t_0} \neq 0$$

for any value of t_0.

Example 3.2.7. Verify that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions of the following differential equation

$$2t^2 y'' + 3ty' - y = 0 \quad t > 0.$$

Compute the Wronskian of y_1 and y_2 at t_0.

Solution. Since we have

$$y_1'(t) = \frac{1}{2t^{1/2}} \quad \text{and} \quad y_1''(t) = -\frac{1}{4t^{3/2}},$$

we have

$$2t^2\left(-\frac{1}{4t^{3/2}}\right) + 3t\left(\frac{1}{2t^{1/2}}\right) - t^{1/2} = \left(-\frac{1}{2} + \frac{3}{2} - 1\right)t^{1/2} = 0.$$

Similarly,

$$y_2'(t) = -t^{-2} \quad \text{and} \quad y_2''(t) = 2t^3,$$

we have

$$2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} = (4 - 3 - 1)t^{-1} = 0.$$

Next, we calculate the Wronskian

$$W[y_1, y_2; t_0] = \begin{vmatrix} t_0^{1/2} & t_0^{-1} \\ t_0^{-1/2}/2 & -t_0^{-2} \end{vmatrix} = -\frac{3}{2}t_0^{-3/2} \neq 0.$$

Consequently, we conclude that y_1 and y_2 form a fundamental set of solutions. Thus, the general solution of the differential equation is

$$y(t) = c_1 t^{1/2} + c_2 t^{-1}, \quad t > 0.$$

□

We summarize the discussion in this section as follows: to find the general solution of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad \alpha < t < \beta,$$

- first find two functions y_1 and y_2 such that they satisfy the differential equation in $\alpha < t < \beta$.
- Compute the Wronskian $W[y_1, y_2; t]$ and make sure that there is a point in the interval such that the Wronskian is nonzero.
- Under these circumstances y_1 and y_2 form a fundamental set of solutions. The general solution is

  $$y(t) = c_1 y_1(t) + c_2 y_2(t),$$

  where c_1 and c_2 are arbitrary constants.
- Furthermore, if initial conditions are given at a given point in $\alpha < t < \beta$, then c_1 and c_2 are chosen so as to satisfy these conditions.

3.3 Complex Roots of the Characteristic Equation

Keywords: Complex Conjugate Roots, Euler's Formula

In this section, we continue our discussion of the second-order linear differential equation:

$$ay'' + by' + cy = 0, \tag{3.17}$$

where a, b, and c are given real numbers. In Section 3.1, we found that if we seeks solution of the form $y = e^{rt}$, then r must be a root of the characteristic equation:

$$ar^2 + br + c = 0. \tag{3.18}$$

We denote r_1 and r_2 the roots of (3.18). We dealt with the case when $r_1 \neq r_2$ in Section 3.1. In this section, we discuss the case when r_1 and r_2 are complex conjugates. When $b^2 - 4ac < 0$, the roots of (3.18) are conjugate complex numbers. Thus, we may denote

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu, \tag{3.19}$$

where λ and μ are real and i is the imaginary number such that $i^2 = -1$. In this case, the corresponding expressions for y_1 and y_2 are

$$\begin{aligned} y_1(t) &= \exp\left([\lambda + i\mu]t\right) = e^{\lambda t} e^{i\mu t} = e^{\lambda t}\left(\cos(\mu t) + i\sin(\mu t)\right), \\ y_2(t) &= \exp\left([\lambda - i\mu]t\right) = e^{\lambda t} e^{-i\mu t} = e^{\lambda t}\left(\cos(\mu t) - i\sin(\mu t)\right), \end{aligned} \tag{3.20}$$

where we have made use of the **Euler's formula**:

$$\begin{aligned} e^{it} &= \cos t + i\sin t, \\ e^{-it} &= \cos t - i\sin t. \end{aligned} \tag{3.21}$$

The following result is fundamental in dealing with differential equations with complex-valued solutions.

Theorem 3.3.1. Consider the second-order linear differential equation:

$$y'' + p(t)y' + q(t)y = 0, \tag{3.22}$$

where p and q are continuous real-valued functions. If $y(t) = u(t) + iv(t)$ is a complex-valued solution of (3.22), then its real part $u(t)$ and imaginary part $v(t)$ are also solutions of this equations. Conversely, if two real-valued functions $u(t)$ and $v(t)$ satisfy (3.22), then the complex-valued function $y = u + iv$ is also a solution of (3.22).

For $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, we can also choose

$$\begin{aligned} u(t) &= e^{\lambda t}\cos(\mu t), \\ v(t) &= e^{\lambda t}\sin(\mu t), \end{aligned} \tag{3.23}$$

to form a fundamental set of solutions of (3.17) since the Wronskian $W[u, v; t]$ is

$$W[u, v; t] = \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} = \begin{vmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ -\mu e^{\lambda t}\sin(\mu t) & \mu e^{\lambda t}\cos(\mu t) \end{vmatrix} = \mu e^{2\lambda t}.$$

In this case, the general solution of (3.17) can be written as

$$y(t) = c_1 u(t) + c_2 v(t) = e^{\lambda t}(c_1\cos(\mu t) + c_2\sin(\mu t)).$$

Example 3.3.2. Find the general solution of the differential equation

$$y'' + y' + 9.25y = 0. \tag{3.24}$$

Also find the solution that satisfies the initial conditions:

$$y(0) = 2, \quad y'(0) = 8. \tag{3.25}$$

Solution. First, we solve the characteristic equation of (3.24):

$$r^2 + r + 9.25 = 0$$

so its roots are

$$r_1 = -\frac{1}{2} + 3i, \quad r_2 = -\frac{1}{2} - 3i.$$

Therefore, two solutions of (3.24) are

$$\begin{aligned} y_1(t) &= e^{-t/2}\left(\cos(3t) + i\sin(3t)\right), \\ y_2(t) &= e^{-t/2}\left(\cos(3t) - i\sin(3t)\right). \end{aligned} \tag{3.26}$$

The Wronskian of y_1 and y_2 is

$$W[y_1, y_2; t] = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = -6ie^{-t} \neq 0.$$

Hence, the general solution of (3.24) can be expressed as $y(t) = c_1 y_1(t) + c_2(t)$. However, the initial-value problem (3.24) and (3.25) has only real coefficients, and it is often desirable to express the solution in terms of real-valued functions. To do this, we know from Theorem 3.3.1 that $u(t) = e^{-t/2}\cos(3t)$ and $v(t) = e^{-t/2}\sin(3t)$ are the solutions of (3.24). We find that

$$W[u, v; t] = \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} = 3e^{-t}\left(\cos^2(3t) - (-\sin^2(3t))\right) = 3e^{-t} \neq 0.$$

Therefore, $u(t)$ and $v(t)$ form a fundamental set of solutions of (3.24), and the general solution can be written as

$$y(t) = c_1 u(t) + c_2 v(t) = e^{-t/2}\left(c_1\cos(3t) + c_2\sin(3t)\right),$$

where c_1 and c_2 are arbitrary. Applying the initial conditions, we obtain

$$c_1 = 2 \quad \text{and} \quad -\frac{1}{2}c_1 + 3c_2 = 8.$$

Therefore, we have $c_1 = 2$ and $c_2 = 3$. The solution of the initial-value problem (3.24) and (3.25) is

$$y(t) = e^{-t/2}\left(2\cos(3t) + 3\sin(3t)\right).$$

□

Example 3.3.3. Find the solution of the initial-value problem

$$16y'' - 8y' + 145y = 0, \quad y(0) = -2, \quad y'(0) = 1. \tag{3.27}$$

Solution. The characteristic equation is

$$16r^2 - 8r + 145 = 0$$

and its roots are

$$r_1 = \frac{1}{4} + 3i, \quad r_2 = \frac{1}{4} - 3i.$$

Thus, the general solution is

$$y(t) = e^{t/4}\left(c_1\cos(3t) + c_2\sin(3t)\right).$$

Applying the first initial condition, we set $t = 0$ and obtain

$$y(0) = c_1 = -2.$$

For the second initial condition, we differentiate the general solution and set $t = 0$. In this way, we obtain

$$y'(0) = \frac{1}{4}c_1 + 3c_2 = 1 \implies c_2 = \frac{1}{2}.$$

Therefore, the solution of the initial-value problem (3.27) is

$$y(t) = -2e^{t/4}\cos(3t) + \frac{1}{2}e^{t/4}\sin(3t).$$

□

Example 3.3.4. Find the general solution of

$$y'' + 9y = 0.$$

Solution. The characteristic equation is $r^2 + 9 = 0$ with its roots $r_1 = 3i$ and $r_2 = -3i$. Thus, the general solution is

$$y(t) = c_1\cos(3t) + c_2\sin(3t).$$

□

3.4 Repeated Roots; Reduction of Order

Keywords: Method of Reduction of Order

In this section, we continue our discussion of the second-order linear differential equation:

$$ay'' + by' + cy = 0, \tag{3.28}$$

where a, b, and c are given real numbers. In Section 3.1, we found that if we seeks solution of the form $y = e^{rt}$, then r must be a root of the characteristic equation:

$$ar^2 + br + c = 0. \tag{3.29}$$

We denote r_1 and r_2 the roots of (3.18). We dealt with the case when $r_1 \neq r_2$ in Section 3.1; we also discussed the case when r_1 and r_2 are complex conjugates in Section 3.3. In this section, we discuss the case when $r_1 = r_2$.

In the case when $r_1 = r_2$, we must have

$$b^2 - 4ac = 0 \quad \text{and} \quad r_1 = r_2 = -\frac{b}{2a}.$$

It immediate yields the same solution

$$y_1(t) = e^{-bt/(2a)}$$

of the differential equation (3.28). To find the second solution, we assume that

$$y(t) = v(t)y_1(t) = v(t)e^{-bt/(2a)}.$$

We find the formula of $v(t)$. This procedure is called the method of **reduction of order**. Substituting for y in (3.28) to determine $v(t)$, we have

$$y'(t) = v'(t)e^{-bt/(2a)} - \frac{b}{2a}v(t)e^{-bt/(2a)},$$

$$y''(t) = v''(t)e^{-bt/(2a)} - \frac{b}{a}v'(t)e^{-bt/(2a)} + \frac{b^2}{4a^2}v(t)e^{-bt/(2a)}.$$

Then, by substituting in (3.28), we have

$$\left[a\left(v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\right) + b\left(v'(t) - \frac{b}{2a}v(t)\right) + cv(t)\right]e^{-bt/(2a)} = 0.$$

Cancelling the factor $e^{-bt/(2a)}$, which is nonzero, and rearranging the remaining terms, we have

$$av''(t) + (-b+b)v'(t) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v(t) = 0.$$

The term involving $v'(t)$ is obviously zero. Further, the coefficient of $v(t)$ is

$$c - \frac{b^2}{4a} = \frac{4ac - b^2}{4a} = 0.$$

Hence, it reduces to

$$v''(t) = 0 \iff v(t) = c_1 + c_2 t.$$

As a result, we have

$$y(t) = c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)}.$$

Thus, y is a linear combination of the two solutions:

$$y_1(t) = e^{-bt/(2a)} \quad \text{and} \quad y_2(t) = te^{-bt/(2a)}.$$

The Wronskian $W[y_1, y_2; t]$ is

$$W[y_1, y_2; t] = \begin{vmatrix} e^{-bt/(2a)} & te^{-bt/(2a)} \\ -\frac{b}{2a}e^{-bt/(2a)} & \left(1 - \frac{bt}{2a}\right)e^{-bt/(2a)} \end{vmatrix} = e^{-bt/a} \neq 0.$$

Then, y_1 and y_2 form a fundamental set of solutions of (3.28) and the general solution of (3.28) is

$$y(t) = c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)}. \tag{3.30}$$

Example 3.4.1. Find the solution of the initial-value problem:

$$y'' - y' + \frac{y}{4} = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{3}. \tag{3.31}$$

Solution. The characteristic equation is

$$r^2 - r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2 = 0.$$

The roots are $r_1 = r_2 = 1/2$. Thus, the general solution of (3.31) is

$$y(t) = c_1 e^{t/2} + c_2 t e^{t/2}.$$

Applying the first initial condition, we have

$$y(0) = c_1 = 2.$$

Applying the second initial condition, we differentiate the general solution and set $t = 0$. Thus, we obtain

$$y'(0) = \frac{1}{2}c_1 + c_2 = \frac{1}{3} \implies c_2 = -\frac{2}{3}.$$

Hence, the solution of the initial-value problem (3.31) is

$$y(t) = 2e^{t/2} - \frac{2}{3}te^{t/2}.$$

□

Example 3.4.2. Find the solution of the initial-value problem:

$$9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1. \tag{3.32}$$

Solution. The characteristic equation is $9r^2 - 12r + 4 = 0$ with roots being

$$r = \frac{12 \pm \sqrt{12^2 - 4 \cdot 9 \cdot 4}}{18} = \frac{2}{3}.$$

The general solution is

$$y(t) = c_1 e^{2t/3} + c_2 t e^{2t/3} \implies y'(t) = \frac{2}{3} c_1 e^{2t/3} + c_2 \left(\frac{2}{3}t + 1\right) e^{2t/3}.$$

Using the initial condition, we have

$$c_1 = 2, \quad \frac{2}{3} c_1 + c_2 = -1 \implies c_1 = 2, \quad c_2 = -\frac{5}{3}.$$

Hence, the solution to the IVP is

$$y(t) = 2e^{2t/3} - \frac{5}{3} t e^{2t/3}.$$

□

Example 3.4.3. Find the solution of the initial-value problem:

$$y'' + 4y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = 1. \tag{3.33}$$

Solution. The characteristic equation is $r^2 + 4r + 4 = 0$ with roots being

$$r = -2.$$

The general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} \implies y'(t) = -2c_1 e^{-2t} + c_2 (1 - 2t) e^{-2t}.$$

Using the initial condition, we have

$$c_1 = 2, \quad -2c_1 + c_2 = -1 \implies c_1 = 2, \quad c_2 = 3.$$

Hence, the solution to the IVP is

$$y(t) = 2e^{-2t} + 3te^{-2t}.$$

□

We summarize the results that we have obtained for second-order linear homogeneous equations with constant coefficients $ay'' + by' + cy = 0$. Let r_1 and r_2 be the roots of the corresponding characteristic equation $ar^2 + br + c = 0$.

- If $r_1 \neq r_2$, then the general solution is
$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$
- If r_1 and r_2 are complex conjugates with $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, then the (real) general solution is
$$y(t) = e^{\lambda t}\left(c_1 \cos(\mu t) + c_2 \sin(\mu t)\right).$$
- If $r_1 = r_2 = r$, then the general solution is
$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$
- c_1 and c_2 are determined by initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$, where t_0, y_0, and y_0' are given.

It is worth noting that the method of **reduction order** can be applied to more general case that if we know that y_1 is a solution of the second-order linear differential equation, then we can set
$$y(t) = v(t)y_1(t)$$
to find out the second solution. In fact, if $y_1(t)$ solves the linear second order equation
$$P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0,$$
then substituting $y(t) = v(t)y_1(t)$ will give
$$\begin{aligned}
P(v''y_1 + 2v'y_1' + vy_1'') + Q(v'y_1 + vy_1') + R(vy_1) = 0 \\
\Longrightarrow\ y_1Pv'' + (2y_1'P + y_1Q)v' + (Py_1'' + Qy_1' + Ry_1)v = 0 \\
\Longrightarrow\ y_1Pv'' + (2y_1'P + y_1Q)v' = 0
\end{aligned}$$
since $Py_1'' + Qy_1' + Ry_1 = 0$. There must be no term involving v.

Example 3.4.4. Given that $y_1(t) = t^{-1}$ is a solution of
$$2t^2y'' + 3ty' - y = 0 \quad t > 0. \tag{3.34}$$
Find a fundamental set of solutions.

Solution. We set $y(t) = v(t)t^{-1}$. Then, we have
$$y'(t) = v'(t)t^{-1} - v(t)t^{-2}, y''(t) \quad = v''(t)t^{-1} - 2v'(t)t^{-1} + 2v(t)t^{-3}.$$
Substituting for y, y', and y'' in (3.34) and collecting terms, we obtain
$$\begin{aligned}
2t^2\left(v''(t)t^{-1} - 2v'(t)t^{-1} + 2v(t)t^{-3}\right) + 3t\left(v'(t)t^{-1} - v(t)t^{-2}\right) - vt^{-1} = 0 \\
\Longrightarrow\ 2tv''(t) + (-4+3)v'(t) + (4t^{-1} - 3t^{-1} - t^{-1})v(t) = 0 \\
\Longrightarrow\ 2tv''(t) - v'(t) = 0.
\end{aligned} \tag{3.35}$$

If we let $w = v'$, then the second-order differential equation (3.35) becomes

$$2tw'(t) - w = 0.$$

Separating the variables and solving for $w(t)$, we find that

$$w(t) = v'(t) = c_1 t^{1/2}.$$

Then, one final integration yields

$$v(t) = \frac{2}{3} c_1 t^{3/2} + c_2.$$

If follows that

$$y(t) = v(t)t^{-1} = \frac{2}{3} c_1 t^{1/2} + c_2 t^{-1}$$

where c_1 and c_2 are arbitrary constants. The second term on the right-hand side is a multiple of y_1 and can be dropped, but the first term provides a new solution $y_2(t) = t^{1/2}$. One can verify that

$$W[y_1, y_2; t] = \frac{3}{2} t^{-3/2} \neq 0 \quad \text{for } t > 0.$$

Consequently, $y_1(t) = t^{-1}$ and $y_2(t) = t^{1/2}$ form a fundamental set of solutions of (3.34). □

3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

Keywords: Nonhomogeneous, Particular Solution, Undetermined Coefficiets

In this section, we study the nonhomogeneous equation:

$$y'' + p(t)y' + q(t)y = g(t). \tag{3.36}$$

The corresponding homogeneous equation reads:

$$y'' + p(t)y' + q(t)y = 0 \tag{3.37}$$

Observe that

- If $Y_1(t)$ and $Y_2(t)$ are two solutions of (3.36), then $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation (3.37).
- Assume that the general solution of (3.37) is given by $c_1 y_1(t) + c_2 y_2(t)$ for some other functions y_1 and y_2 with c_1 and c_2 being arbitrary, then the general solution of the nonhomogeneous equation (3.36) can be written as

  $$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

  where $Y(t)$ is any solution of the nonhomogeneous equation (3.36). We call $Y(t)$ to be a **particular solution**.
- In general, to solve the nonhomogeneous equation (3.36), we have to first find the general solution of the corresponding homogeneous equation (3.37), and find a particular solution of (3.36).

In the following examples, we introduce techniques for finding particular solutions using the method of **undetermined coefficients**.

Example 3.5.1. Find a particular solution of the following nonhomogeneous equation:

$$y'' - 3y' - 4y = 3e^{2t}. \tag{3.38}$$

Solution. We seek a particular solution Y such that $Y'' - 3Y' - 4Y$ is equal to $3e^{2t}$. Since the exponential function reproduces itself through differentiation, the most plausible way to achieve the desired result is to assume that $Y(t)$ is some multiple of e^{2t}, that is, we assume that

$$Y(t) = Ae^{2t}$$

where A is a coefficient to be determined. Then, we have

$$Y''(t) = 4Ae^{2t} \quad \text{and} \quad Y'(t) = 2Ae^{2t}.$$

Substituting these expressions into (3.38), we have

$$Y'' - 3Y' - 4Y = (4A - 6A - 4A)e^{2t} = 3e^{2t} \implies -6A = 3 \implies A = -\frac{1}{2}.$$

Thus, a particular solution is

$$Y(t) = -\frac{1}{2}e^{2t}.$$

□

Example 3.5.2. Find a particular solution of the following nonhomogeneous equation:

$$y'' - 3y' - 4y = 2\sin(t). \tag{3.39}$$

Solution. Since the right-hand side is $2\sin(t)$, the differentiation of $\sin(t)$ and $\cos(t)$ will produce the trigonometric functions. As a result, we can assume that the particular solution $Y(t)$ is

$$Y(t) = A\sin(t) + B\cos(t)$$

for some coefficients A and B to be determined. By calculation, we have

$$Y'(t) = A\cos(t) - B\sin(t), \quad Y''(t) = -A\sin(t) - B\cos(t).$$

Substituting these into (3.39), we have

$$Y'' - 3Y' - 4Y = (-A + 3B - 4A)\sin(t) + (-B - 3A - 4B)\cos(t) = 2\sin(t).$$

Then, we have

$$-5A + 3B = 2, \quad -3A - 5B = 0.$$

Therefore, we have

$$A = -\frac{5}{17}, \quad B = \frac{3}{17}.$$

The particular solution of (3.39) is

$$Y(t) = -\frac{5}{17}\sin(t) + \frac{3}{17}\cos(t).$$

□

Example 3.5.3. Find a particular solution of the following nonhomogeneous equation:

$$y'' - 3y' - 4y = -8e^{t}\cos(2t). \tag{3.40}$$

Solution. In this case, we assume $Y(t)$ is the product of e^t and a linear combination of $\cos(2t)$ and $\sin(2t)$. That is, we assume that

$$Y(t) = Ae^t \cos(2t) + Be^t \sin(2t).$$

Then, we have

$$Y'(t) = (A + 2B)e^t \cos(2t) + (-2A + B)e^t \sin(2t)$$

and

$$Y''(t) = (-3A + 4B)e^t \cos(2t) + (-4A - 3B)e^t \sin(2t).$$

By substituting these into (3.40), we find that A and B satisfy

$$10A + 2B = 8, \quad 2A - 10B = 0.$$

Hence, we have

$$A = \frac{10}{13}, \quad B = \frac{2}{13}.$$

The particular solution of (3.40) is

$$Y(t) = \frac{10}{13}e^t \cos(2t) + \frac{2}{13}e^t \sin(2t).$$

□

Suppose that the right-hand side $g(t)$ in (3.36) is the sum of two terms, that is, $g(t) = g_1(t) + g_2(t)$, and suppose that Y_1 and Y_2 are solutions of the equations

$$y'' + p(t)y' + q(t)y = g_1(t)$$

and

$$y'' + p(t)y' + q(t)y = g_2(t)$$

respectively. Then, $Y_1 + Y_2$ is a solution of the equation

$$y'' + p(t)y' + q(t)y = g(t).$$

For instance, to find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin(t) - 8e^t \cos(2t),$$

we just add those particular solutions found in previous example to obtain a particular solution:

$$Y(t) = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin(t) + \frac{3}{17}\cos(t) + \frac{10}{13}e^t \cos(2t) + \frac{2}{13}e^t \sin(2t).$$

Example 3.5.4. Find a particular solution of

$$y'' - 3y' - 4y = 2e^{-t}. \tag{3.41}$$

Solution. If we assume that the particular solution is of them form $Y(t) = Ae^{-t}$, then we have

$$Y'' - 3Y' - 4Y = (A + 3A - 4A)e^{-t} = 2e^{-t}.$$

Since the left-hand side of (3.41) is zero in this case, there is no choice of A for which $0 = 2e^{-t}$. Therefore, there is no particular solution of the assumed form. The reason for this is that $\phi(t) = e^{-t}$ is a solution of the corresponding homogeneous equation $y'' - 3y' - 4y = 0$.

Instead, we can assume that $Y(t) = Ate^{-t}$. Then, we have

$$Y'(t) = Ae^{-t} - Ate^{-t}, \quad Y''(t) = -2Ae^{-t} + Ate^{-t}.$$

Substituting these expression into (3.41), we have

$$Y'' - 3Y' - 4Y = (-2A - 3A)e^{-t} + (A + 3A - 4A)te^{-t} = 2e^{-t}.$$

The coefficient of te^{-t} is zero, and from the terms involving e^{-t}, we have $-5A = 2$ and so $A = -2/5$. Thus, a particular solution of (3.41) is

$$Y(t) = -\frac{2}{5}te^{-t}.$$

□

We remark that if the assumed form of the particular solution duplicates a solution of the corresponding homogeneous equation, then modify the assumed particular solution by multiplying it by t. Sometimes this modification will be insufficient to remove all duplication with the solutions of the homogeneosu equation, in which case it is necessary to multiply by t a second time. For a second-order equation, it will never be necessary to carry the process further than this.

We summarize the steps for solving

$$ay'' + by' + cy = g(t). \tag{3.42}$$

- Find a general solution of the corresponding homogeneous equation $ay'' + by' + cy = 0$. Denote $c_1y_1(t) + c_2y_2(t)$ its general solution.
- Make sure that the function $g(t)$ is of the form of exponential functions, sines, cosines, polynomial, or sums or products of such functions. If this is not the case, use the method of variation of parameters (in Section 3.6).
- If $g(t) = g_1(t) + \cdots + g_n(t)$, then, we find the particular solution $Y_i(t)$ to the following subproblem
$$ay'' + by' + cy = g_i(t)$$
for any $i = 1, \cdots, n$. The particular solution of (3.42) is $Y(t) = Y_1(t) + \cdots + Y_n(t)$.
- The general solution of the nonhomogeneous equation is the sum of the particular solution and the general solution of the homogeneous one. That is, the general solution of (3.42) is
$$y(t) = Y(t) + c_1y_1(t) + c_2y_2(t).$$
- When initial conditions are provided, use them to determine the values of the arbitrary constants remaining in the general solution.

3.6 Variation of Parameters

Keywords: variation of parameters, nonhomogeneous equations, Wronskian

In this section, we introduce the method of **variation of parameters**, to solve the second-order linear differential equation. This is a general method and it can be applied to any equation, and it requires no detailed assumptions about the form of the solution.

Example 3.6.1. Find the general solution of the following equation:

$$y'' + 4y = 8\tan t \quad -\frac{\pi}{2} < t < \frac{\pi}{2}. \tag{3.43}$$

Solution. First, we solve the corresponding homogeneous equation $y'' + 4y = 0$. The characteristic equation is

$$r^2 + 4 = 0 \implies r = \pm 2i.$$

That is, the roots of the characteristic equation is $r_1 = 2i$ and $r_2 = -2i$. Hence, the general solution of the homogeneous equation is

$$c_1 \cos(2t) + c_2 \sin(2t).$$

Next, we find the particular solution of (3.43). The basic idea in the method of **variation of parameters** is similar to the method of reduction of order introduced in Section 3.4. We assume that the solution of (3.43) has the form

$$y(t) = u_1(t)\cos(2t) + u_2(t)\sin(2t) \tag{3.44}$$

where the coefficients $u_1(t)$ and $u_2(t)$ are functions of t. We have to determine two functions $u_1(t)$ and $u_2(t)$. We need one more condition to determine $u_1(t)$ and $u_2(t)$ since currently we just have the original equation (3.43). From the formula (3.44), we have

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t) + u_1'(t)\cos(2t) + u_2'(t)\sin(2t).$$

We require that

$$u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0.$$

It then follows that

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t).$$

From this formula, we calculate y'':

$$y''(t) = -4u_1(t)\cos(2t) - 4u_2(t)\sin(2t) - 2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t).$$

Then, substituting for y and y'' into the equation (3.43), we find that

$$\begin{aligned} y'' + 4y &= -4u_1(t)\cos(2t) - 4u_2(t)\sin(2t) - 2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) + 4u_1(t)\cos(2t) + 4u_2(t)\sin(2t) \\ &= -2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) = 8\tan t. \end{aligned}$$

To summarize, the functions $u_1(t)$ and $u_2(t)$ satisfy

$$\begin{aligned} u_1'(t)\cos(2t) + u_2'(t)\sin(2t) &= 0, \\ -2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) &= 8\tan t. \end{aligned} \tag{3.45}$$

From the first equation in (3.45), we have

$$u_2'(t) = -u_1'(t)\frac{\cos(2t)}{\sin(2t)}.$$

Substituting this into the second equation and simplifying it, we obtain

$$u_1'(t) = -\frac{8\tan t\sin(2t)}{2} = -8\sin^2 t = 4\cos(2t) - 4.$$

Here, we have used the fact that

$$\sin(2t) = 2\sin(t)\cos(t) \quad \text{and} \quad \sin^2 t = \frac{1-\cos(2t)}{2}.$$

It implies that

$$u_1'(t) = 2\sin(2t) - 4t + c_1$$

where c_1 is an arbitrary constant. We solve $u_2'(t)$ to obtain

$$u_2'(t) = \frac{8\sin^2 t\cos(2t)}{\sin(2t)} = 4\frac{\sin t(2\cos^2 t - 1)}{\cos t} = 4\sin t\left(2\cos t - \frac{1}{\cos t}\right) = 4\sin(2t) - \frac{4\sin t}{\cos t}.$$

Here, we have used the fact that $\sin(2t) = 2\sin t\cos t$. Having obtained $u_1'(t)$ and $u_2'(t)$, we next integrate so as to find $u_1(t)$ and $u_2(t)$:

$$\begin{aligned} u_1(t) &= \int(4\cos(2t) - 4)dt + c_1 = 2\sin(2t) - 4t + c_1, \\ u_2(t) &= \int\left(4\sin(2t) - \frac{4\sin t}{\cos t}\right)dt + c_2 = -2\cos(2t) + 4\ln(\cos(t)) + c_2. \end{aligned}$$

As a result, the general solution of (3.43) is

$$\begin{aligned} y(t) &= u_1(t)\cos(2t) + u_2(t)\sin(2t) \\ &= 2\sin(2t)\cos(2t) - 4t\cos(2t) + 4\ln(\cos t)\sin(2t) - 2\cos(2t)\sin(2t) + c_1\cos(2t) + c_2\sin(2t) \\ &= -4t\cos(2t) + 4\ln(\cos t)\sin(2t) + c_1\cos(2t) + c_2\sin(2t). \end{aligned}$$

We remark that the terms involving c_1 and c_2 form the general solution of the corresponding homogeneous equation, while the other terms form a particular solution of the nonhomogeneous equation (3.43). □

Example 3.6.2. Find the general solution of the following equation:

$$y'' - 2y' + y = \frac{e^t}{1+t^2}. \tag{3.46}$$

Solution. The characteristic equation of the corresponding homogeneous equation is $r^2 - 2r + 1 = (r-1)^2 = 0$. Thus, the general solution of the homogeneous equation is

$$c_1 e^t + c_2 t e^t.$$

We assume that the general solution of (3.46) is

$$y(t) = u_1(t)e^t + u_2(t)te^t.$$

Then, we have

$$y'(t) = u_1 e^t + (t+1)u_2(t)e^t + u_1'(t)e^t + u_2'(t)e^t.$$

We require that

$$u_1'(t)e^t + u_2'(t)te^t = 0 \implies u_1'(t) = -tu_2'(t).$$

As a result, we have

$$y'(t) = u_1(t)e^t + (t+1)u_2(t)e^t$$

and

$$y''(t) = u_1(t)e^t + (t+2)u_2(t)e^t + u_1'(t)e^t + (t+1)u_2'(t)e^t = u_1(t)e^t + (t+2)u_2(t)e^t + u_2'(t)e^t$$

Substituting these expression into (3.46), we have

$$\begin{aligned} y'' - 2y' + y &= u_1(t)e^t + (t+2)u_2(t)e^t + u_2'(t)e^t - 2u_1(t)e^t - 2(t+1)u_2(t)e^t + u_1(t)e^t + u_2(t)te^t \\ &= u_2'(t)e^t = \frac{e^t}{1+t^2}. \end{aligned}$$

Hence, we have

$$u_1'(t) = -tu_2'(t), \quad u_2'(t) = \frac{1}{1+t^2}.$$

As a result, we have

$$\begin{aligned} u_1(t) &= \int \left(-\frac{t}{1+t^2}\right) dt + c_1 = -\frac{1}{2}\ln(1+t^2) + c_1, \\ u_2(t) &= \int \left(\frac{1}{1+t^2}\right) dt + c_2 = \arctan t + c_2. \end{aligned}$$

Hence, the general solution of (3.46) is

$$y(t) = -\frac{1}{2}e^t \ln(1+t^2) + te^t \arctan t + c_1 e^t + c_2 te^t.$$

□

In general, for the second order (linear) nonhomogeneous equation (NHE) as follows:

$$y'' + p(t)y' + q(t)y = g(t), \tag{3.47}$$

where p, q, and g are given continuous functions of t. Assume that we have a fundamental set of solutions $\{y_1, y_2\}$ to the corresponding homogeneous equation (HE) $y'' + p(t)y' + q(t)y = 0$ and thus the general solution of the HE can be written as

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t),$$

where c_1 and c_2 are arbitrary constants. This is a major assumption. The crucial idea of the variation of parameters is to assume that the solution of the NHE has the form

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t). \tag{3.48}$$

Then we try to determine u_1 and u_2 so that (3.48) solves the NHE. From the formula (3.48), we obtain y' as follows:

$$y'(t) = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t).$$

As in the previous examples, we set

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0. \tag{3.49}$$

As a result, we have

$$y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t). \tag{3.50}$$

Further by differentiating again, we obtain

$$y''(t) = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t).$$

Substituting back to the original NHE, we have

$$\begin{aligned} &u_1(t)\left(y_1''(t)+p(t)y_1'(t)+q(t)y_1(t)\right) \\ &+u_2(t)\left(y_2''(t)+p(t)y_2'(t)+q(t)y_2(t)\right)+u_1'(t)y_1'(t)+u_2'(t)y_2'(t)=g(t). \end{aligned} \tag{3.51}$$

Note that y_1 and y_2 solve the associated HE, hence the equation (3.51) reduces to

$$u_1'y_1'(t)+u_2'(t)y_2'(t)=g(t).$$

Overall, we obtain a system of equation for u_1' and u_2':

$$\begin{aligned} u_1'(t)y_1(t)+u_2'(t)y_2(t)&=0, \\ u_1'(t)y_1'(t)+u_2'(t)y_2'(t)&=g(t). \end{aligned} \tag{3.52}$$

Solving the system we obtain

$$u_1'(t)=-\frac{y_2(t)g(t)}{|W[y_1,y_2](t)|} \quad \text{and} \quad u_2'(t)=\frac{y_1(t)g(t)}{|W[y_1,y_2](t)|},$$

where $W[y_1,y_2](t)$ is the Wronskian matrix of y_1 and y_2 with nonzero determinant. Hence, we have

$$u_1(t)=-\int\frac{y_2(t)g(t)}{|W[y_1,y_2](t)|}dt+c_1 \quad \text{and} \quad u_2(t)=\int\frac{y_1(t)g(t)}{|W[y_1,y_2](t)|}dt+c_2.$$

Consequently, the solution of the NHE is

$$y(t)=-y_1(t)\int\frac{y_2(t)g(t)}{|W[y_1,y_2](t)|}dt+y_2(t)\int\frac{y_1(t)g(t)}{|W[y_1,y_2](t)|}dt+c_1y_1(t)+c_2y_2(t).$$

3.7 Mechanical and Electrical Vibrations

Keywords: Newton's second law, simple harmonic motion, damping, spring-mass system, simple electric circuit

In this section, we discuss some applications of second-order differential equations in physics. The second-order differential equations with constant coefficients can be used to describe many important physical processes such as mechanical and electrical oscillations. In the following, we study the motion of a mass on a spring in detail to understand the behavior of vibrating systems. The principles involved are common to many problems.

Consider a mass m hanging at rest of the end of a vertical spring of original length ℓ. The mass causes an elongation L of the spring in the downward (positive direction). In this static situation, there are two forces acting at the point where the mass is attached to the spring: the gravitational force $w=mg$ (with g gravity acceleration) and a force F_s due to the spring that acts upward. According to the *Hooke's law*, the spring force is proportional to L if the elongation L of the spring is small. That is, we write $F_s=-kL$, where the constant k is called the spring constant, and the minus sign is due to the fact that the spring force acts in the upward (negative) direction. Since the mass is in equilibrium, the two forces balance each other, which yields

$$w+F_s=mg-kL=0.$$

For a given weight $w=mg$, we can measure L and the use the above equation to determine k. Note that k has the units of force per unit length.

In the corresponding dynamic problem, we are interested in studying the motion of the mass when it is acted on by an external force or is initially displaced. Let $u(t)$, measured positive in the downward direction, denote the displacement of the mass from its equilibrium position at time t. Then, $u(t)$ is related to the forces acting on the mass through Newton's law of motion:

$$mu''(t) = f(t), \tag{3.53}$$

where u'' is the acceleration of the mass and f is the net force acting on the mass. Observe that both u and f are functions of time. In this dynamic problem there are now four separate forces that must be considered.

1. The weight $w = mg$ of the mass always acts downward.
2. The spring force F_s is assumed to be proportional to the total elongation $L + u$ of the spring and always acts to restore the spring to its natural position. If $L + u > 0$, then the spring is extended, and the spring force is directed upward. In this case,

   $$F_s = -k(L + u).$$

   If $L + u < 0$, then the spring is compressed a distance $|L + u|$, and the spring force, which is now directed downward, is given by $F_s = k\,|L + u|$. However, when $L + u < 0$, it follows that $|L + u| = -(L + u)$, so F_s is again given by the same relation above.
3. The damping force F_d always acts in the direction opposite to the direction of motion of the mass and it is given by

   $$F_d(t) = -\gamma u'(t),$$

   where $\gamma > 0$ is the so-called *damping constant*. This force may arise from several sources: resistance from the air or other medium in which the mass moves, internal energy dissipation due to the extension or compression of the spring, friction between the mass.
4. An applied external force $F(t)$ is directed downward or upward as $F(t)$ is positive or negative. Often the external force is periodic.

Taking account of these forces, we can now rewrite Newton's law as

$$mu''(t) = w + F_s + F_d + F = mg - k(L + u(t)) - \gamma u'(t) + F(t).$$

Since $mg - kL = 0$, we have

$$mu''(t) + \gamma u'(t) + ku(t) = F(t), \tag{3.54}$$

where m, γ, and k are positive. This model is only an approximation model for the displacement $u(t)$. In particular, there are many other ways of describing the damping or spring forces. In the above derivation we have also neglected the mass of the spring in comparison with the mass of the attached body.

The complete formulation of the vibration problem requires that we specify two initial conditions, namely, the initial position u_0 and the initial velocity v_0 of the mass:

$$u(0) = u_0 \quad \text{and} \quad u'(0) = v_0.$$

Example 3.7.1 (Simple Harmonic Motion)**.** Now we study a simplified version of the general vibrational system (3.54). Assume that there is no external force ($F(t) = 0$) and no damping $\gamma = 0$. Then, it becomes

$$mu'' + ku = 0. \tag{3.55}$$

The characteristic equation reads

$$mr^2 + k = 0 \implies r = \pm i\sqrt{\frac{k}{m}} =: \pm i\omega_0, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

The constant ω_0 is the so-called *vibrational frequency* of the system. Then, the general solution to (3.55) is

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$

where A and B are two arbitrary constants determined by the initial conditions. In discussing the solution formula, it is convenient to rewrite the equation in the form

$$u(t) = R\cos(\omega_0 t - \delta), \tag{3.56}$$

where

$$A = R\cos\delta, \quad B = R\sin\delta, \quad R = \sqrt{A^2 + B^2}, \quad \text{and} \quad \tan\delta = \frac{B}{A}.$$

It is easy to verify that if R and δ in (3.56) satisfies

$$R\cos(\omega_0 t - \delta) = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$

using the trigonometric formula $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ (with $\alpha = \omega_0 t$ and $\beta = \delta$), we have

$$u(t) = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

This implies that $A = R\cos\delta$ and $B = R\sin\delta$. Thus, $R = \sqrt{A^2 + B^2}$ and $\tan\delta = B/A$. In calculating δ, we have to choose the correct quadrant; this can be done by checking the signs of $\cos\delta$ and $\sin\delta$. This model describes a periodic (or *simple harmonic*) motion of the mass. The *period* of the motion is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}.$$

The maximum displacement R of the mass from equilibrium is the *amplitude* of the motion. The dimensionless parameter δ is called the *phase*, or phase angle, and it measures the displacement of the wave from its normal position corresponding to $\delta = 0$.

Note that the motion described by (3.56) has a constant amplitude that does not diminish with time. This reflects the fact that, in the absence of damping, there is no way for the system to dissipate the energy imparted to it by the initial displacement and velocity. For a given mass m and spring constant k, the system vibrates at the same frequency ω_0, regardless of the initial conditions. However, the initial conditions do help to determine the amplitude of the motion. The period T increases as the mass increases, so lager masses vibrate more slowly. On the other hand, T decreases as k increases, which means that stiffer springs cause the system to vibrate more rapidly.

Example 3.7.2 (Damped Free Vibrations)**.** We consider the case with no external force but included damping. That is, the governing equation becomes

$$mu'' + \gamma u' + ku = 0. \tag{3.57}$$

The characteristic equation is

$$mr^2 + \gamma r + k = 0$$

and its roots are

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}.$$

Denote r_1 and r_2 the roots with positive and negative signs in the above formula, respectively. Depending on the sign of $\gamma^2 - 4km$, the solution u has one of the forms:

- $\gamma^2 - 4km > 0$, $u = Ae^{r_1 t} + Be^{r_2 t}$;
- $\gamma^2 - 4km = 0$, $u = (A + Bt)e^{-\gamma t/(2m)}$;
- $\gamma^2 - 4km < 0$, $u = e^{-\gamma t/(2m)}\,(A\cos(\mu t) + B\cos(\mu t))$, where $\mu = (2m)^{-1}(4km - \gamma^2)^{1/2} > 0$.

Note that m, γ, and k are positive, we always have $\gamma^2 - 4km < \gamma^2$. Hence, if $\gamma^2 - 4km \geqslant 0$, then the values of r_1 and r_2 are always negative. If $\gamma^2 - 4km < 0$, then the values of r_1 and r_2 are complex with negative real part. Thus, in all cases, the solution $u \to 0$ as $t \to +\infty$; this occurs regardless of the values of the arbitrary constants, i.e., the initial conditions. This confirms our intuitive expectation, namely, that damping gradually dissipates the energy initially imparted to the system, and consequently the motion dies out with increasing time.

Now we study the third case with $\gamma^2 - 4km < 0$. It occurs when the damping is very small. If we let $A = R\cos\delta$ and $B = R\sin\delta$ with appropriate R and δ, we obtain the solution in the following form:

$$u(t) = Re^{-\gamma t/(2m)}\cos(\mu t - \delta).$$

The displacement u lies between $\pm Re^{-\gamma t/(2m)}$; thus it resembles a cosine wave whose amplitude decreases as t increases.

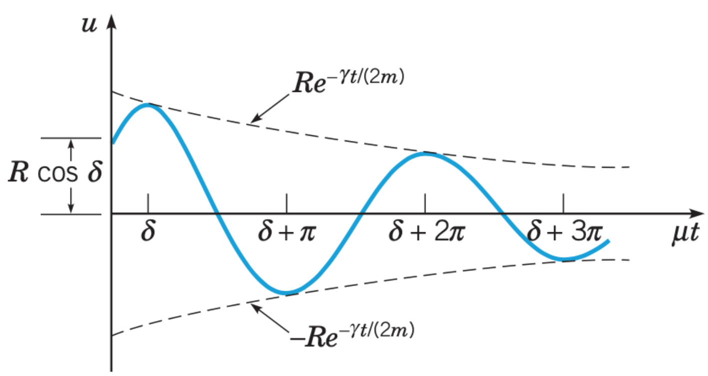


Figure 3.1: Damped vibration without external force.

This motion is called *damped oscillation* or *damped vibration.* The overall amplitude factor depends on m, γ, k, and the initial conditions. Note that this motion is not periodic and the parameter μ determines the frequency with which the mass oscillates back and forth; consequently, μ is called the *quasi-frequency.* By comparing μ with the frequency ω_0 of undamped motion, we obtain

$$\frac{\mu}{\omega_0} = \frac{(2m)^{-1}(4km - \gamma^2)^{1/2}}{(k/m)^{1/2}} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2} \approx 1 - \frac{\gamma^2}{8km}.$$

The last approximation comes from the Taylor expansion and it is valid only when $\gamma^2/4km$ is small; we refer to this situation as *small damping.* Thus the effect of small damping is to reduce slight the frequency of the oscillation. By analogy, we can define the *quasi-period* T_d to be

$$T_d = \frac{2\pi}{\mu}$$

of the motion. It is the time between successive maxima or successive minima of the position of the mass, or between successive passages of the mass through its equilibrium position while going in the same direction. The relation between T_d and T is given by

$$\frac{T_d}{T} = \frac{\omega_0}{\mu} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2} \approx 1 + \frac{\gamma^2}{8km},$$

where again the last approximation is valid in the small damping regime. Thus small damping increases the quasi-period. We remark that it is not the magnitude of γ alone that determines whether damping is large or small, but the magnitude of $\gamma^2/(4km)$. When $\gamma^2/(4km)$ is small, then damping has a small effect on the quasi-frequency and quasi-period of the motion. On the other hand, if we want to study the detailed motion of the mass for all time, then we can *never* neglect the damping force, no matter how small it is.

Another situation can happen when $\gamma^2/(4km)$ increases. As it increases, the quasi-frequency μ decreases and the quasi-period T_d increases. In fact, $\mu \to 0$ and $T_d \to +\infty$ when $\gamma \to 2\sqrt{km}$. As indicated in the solution formulas, the nature of the solution changes as γ passes through the value $2\sqrt{km}$. The motion with $\gamma = 2\sqrt{km}$ is said to be *critically damped*. For larger values of γ with $\gamma > 2\sqrt{km}$, the motion is said to be *overdamped*. In these cases, the mass may pass through its equilibrium position at most once and then creep back to it. The mass does not oscillate about the equilibrium, as it does for small γ.

Example 3.7.3. We consider the model of a simple electric circuit. The current I (measured in Amperes) is a function of time t. The resistance R (in Ohms), the capacitance C (in Farads), and the inductance L (in Henrys) are all positive and are assumed to be known constants. The impressed voltage E (in Volts) is a given function of time t. Another physical quantity that enters the discussion is the total charge Q (in Coulombs) on the capacitor at time t. The relation

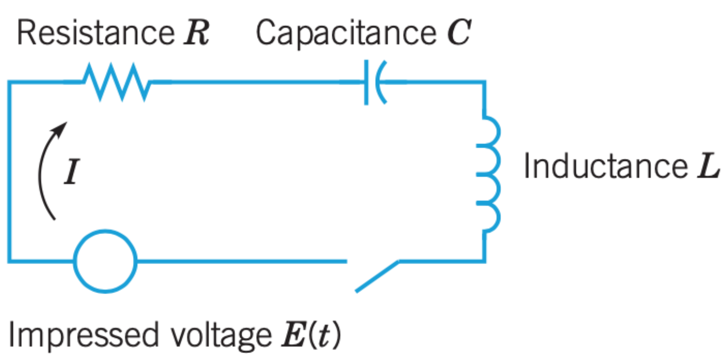


Figure 3.2: A simple electric circuit.

between charge Q and current I is

$$I(t) = Q'(t).$$

The flow of current in the circuit is governed by Kirchhoff's second law: In a closed circuit the impressed voltage is equal to the sum of voltage drops in the rest of the circuit. According to the elementary laws of electricity, we know that (i) the voltage drop across the resistor is RI; (ii) the voltage drop across the capacitor is Q/C; and (iii) the voltage drop across the inductor is $LI'(t)$. Hence, by Kirchhoff's law, we have

$$LI'(t) + RI(t) + \frac{1}{C}Q(t) = E(t).$$

Noting that $I = Q'(t)$, we obtain the differential equation

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t) \tag{3.58}$$

for the charge Q. The initial conditions are

$$Q(t_0) = Q_0 \quad \text{and} \quad Q'(t_0) = I(t_0) = I_0.$$

The model describing this simple electric circuit is again the second-order differential equation with constant coefficients, which is precisely the same form as the one that describes the motion of a spring-mass system. This is a good example of the unifying role of mathematics: once you know how to solve second-order linear equations with constant coefficients, you can interpret the results in terms of mechanical vibrations, electric circuits, or any other physical situation that leads to the same problem.

3.8 Forced Periodic Vibrations

Keywords: transient and steady-state solutions, forcing functions

In this section, we investigate the vibrational system with periodic external force. The behavior of this simple system models that of many oscillatory systems with an external force due, for example, to a motor attached to the system. We first consider the damping case and look later at the idealized special case without any damping.

Example 3.8.1 (Forced Vibrations with Damping)**.** Let the motion of a certain spring-mass system satisfies the differential equation

$$u'' + u' + \frac{5}{4}u = 3\cos t, \tag{3.59}$$

and the initial conditions $u(0) = 2$ and $u'(0) = 3$. We find out the solution to this IVP and describe the behavior of the solution for large t. The characteristic equation reads:

$$r^2 + r + \frac{5}{4} = 0 \iff r = -\frac{1}{2} \pm i.$$

Thus a general solution of the homogeneous equation has the form

$$e^{-t/2}\left(c_1\cos(t) + c_2\sin(t)\right).$$

A particular solution of the equation has the form $U(t) = A\cos t + B\sin t$, where A and B can be found using the method of undetermined coefficients. One can figure out that $A = 12/17$ and $B = 48/17$. Therefore, the particular solution U has the form

$$U(t) = \frac{12}{17}\cos t + \frac{48}{17}\sin t.$$

The general solution of the model problem (3.59) is

$$u(t) = e^{-t/2}\left(c_1\cos(t) + c_2\sin(t)\right) + \frac{12}{17}\cos t + \frac{48}{17}\sin t.$$

Using the initial conditions, we obtain that $c_1 = 22/17$ and $c_2 = 14/17$. Finally, we have the solution to the IVP as follows:

$$u(t) = \underbrace{e^{-t/2}\left(\frac{22}{17}\cos(t) + \frac{14}{17}\sin(t)\right)}_{\textbf{Transient solution}} + \underbrace{\frac{12}{17}\cos(t) + \frac{48}{17}\sin(t)}_{\textbf{Steady-state solution}}. \tag{3.60}$$

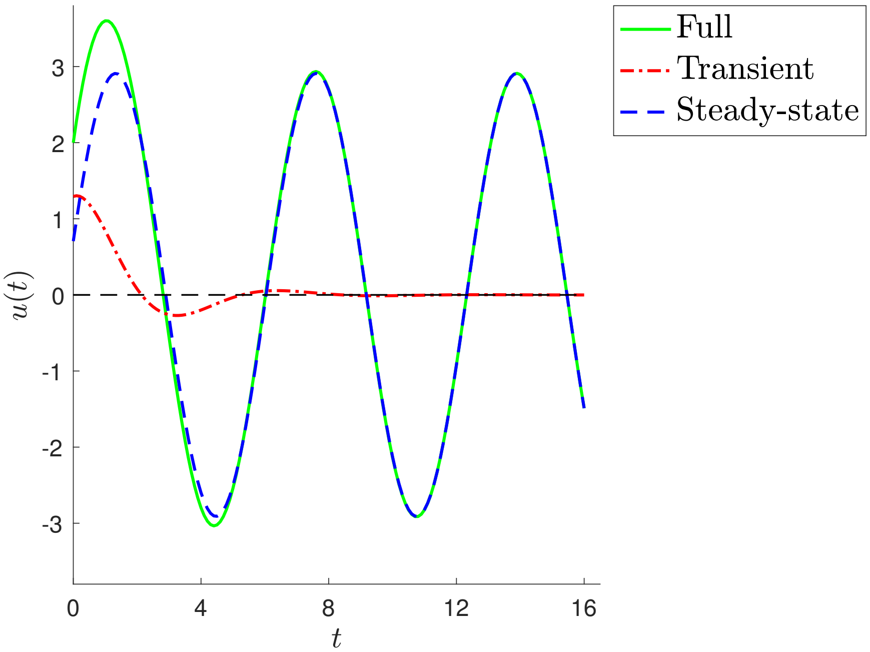


Figure 3.3: Solution profiles of the IVP.

Note that the above solution $u(t)$ consists of two distinct parts: *transient* and *steady-state* solutions. The transient solution contains the exponential factor $e^{-t/2}$ and as a result it rapidly approaches zero. While the stead-state solution involves only sine and cosine functions, so they represent an oscillation that continues indefinitely.

We remark that the transient part comes from the solution of the homogeneous equation and the initial conditions. The steady-state solution is the particular solution of the full nonhomogeneous equation. After a fairly short time, the transient solution is vanishingly small and the full solution is essentially indistinguishable from the steady state.

In general, consider the spring-mass model problem with external force $F(t) \neq 0$ and initial conditions:

$$\begin{aligned} mu''(t) + \gamma u'(t) + ku(t) &= F(t), \\ u(t_0) &= u_0, \\ u'(t_0) &= v_0, \end{aligned} \tag{3.61}$$

where m, γ, k are the mass, damping coefficient, and spring constant. They are all positive. The terms u_0 and v_0 can be interpreted as the initial displacement and initial velocity. Suppose now that the external force is given by some periodic function. For instance, consider the case when $F(t)$ has the following form

$$F(t) = F_0 \cos(\omega t)$$

where F_0 and ω are positive constants representing the amplitude and frequency, respectively, of the force. Combining all we have developed in this course, we know that the solution to the spring-mass system (3.61) in this case has the form as follows:

$$u(t) = \underbrace{c_1 u_1(t) + c_2 u_2(t)}_{u_c(t)} + \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{U(t)}. \tag{3.62}$$

The first two terms (denoting $u_c(t)$ their sum) come from the homogeneous equation corresponding to (3.61) and the latter term $U(t)$ is a particular solution of the full nonhomogeneous equation. The coefficients A and B can be found as usual with the method of undetermined coefficients, while the constants c_1 and c_2 are determined by the initial conditions u_0 and v_0.

The homogeneous solutions u_1 and u_2 depends on the characteristic roots r_1 and r_2 of the equation $mr^2 + \gamma r + k = 0$. Since m, γ, and k are positive, it follows that r_1 and r_2 either are real and negative or are complex conjugates with a negative real part. In either case, both u_1 and u_2 approach zero as $t \to +\infty$. Thus, $u_c(t)$ dies out as t increases, and we called it *transient solution*. In many applications, it is of little importance and (depending on the value of γ) may well be undetectable after only a short period of time.

The remaining term $U(t) = A\cos(\omega t) + B\sin(\omega t)$ does not die out as t increases but persist indefinitely, or as long as the external force is applied. They represent a steady oscillation with the same frequency as the external force and it is called the *steady-state* solution or the *forced response* of the system. The transient solution enables us to satisfy whatever initial conditions u_0 and v_0 are imposed. With increasing time, the energy put into the system by the initial displacement u_0 and velocity v_0 is dissipated through the damping term related to $\gamma u'(t)$, and the motion then becomes the response of the system to the external force $F(t)$. We remark that if there is no damping in the model, the effect of the initial conditions persist for all time (think about the case when $\gamma = 0$).

Example 3.8.2 (Resonance)**.** Now we further study the general spring-mass model (3.61) with periodic external force $F(t) = F_0\cos(\omega t)$. The steady-state solution $U(t)$ can be rewritten as the following form:

$$U(t) = A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta),$$

where $R = \sqrt{A^2 + B^2}$ represents the amplitude and δ satisfying $\tan(\delta) = B/A$ is a phase constant. In fact, those constants A, B, R, and δ can be expressed in terms of m, γ, k, F_0, $\omega_0 = \sqrt{k/m}$, and ω. Note that we can write $k = m\omega_0^2$ and recall that ω_0 is the natural frequency of the unforced system in the absence of damping.

Disclaimer: One may skip the following *tedious* yet *simple* derivation if you are not interested in it. Assume the particular solution $U(t) = A\cos(\omega t) + B\sin(\omega t)$ satisfies the spring-mass system $mu''(t) + \gamma u'(t) + ku(t) = F_0\cos(\omega t)$. Use the method of undetermined coefficients and make use of the table form (see Table 3.1), we obtain the system of equations for A and B:

$$\begin{aligned} (k - m\omega^2)A + \gamma\omega B &= F_0, \\ -\gamma\omega A + (k - m\omega^2)B &= 0. \end{aligned}$$

Note that $k = m\omega_0^2$. Hence, solving the equations, we obtain that

$$A = \frac{m(\omega_0^2 - \omega^2)F_0}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \quad B = \frac{\gamma\omega F_0}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}.$$

Denote $\Delta = \sqrt{m^2(\omega_0^2 - m\omega^2)^2 + \gamma^2\omega^2}$. Then, $A = m(\omega_0^2 - \omega^2)F_0/\Delta^2$ and $B = \gamma\omega F_0/\Delta^2$. Furthermore, using the relation $A = R\cos\delta$ and $B = R\sin\delta$, we obtain that

$$R = \sqrt{A^2 + B^2} = \sqrt{\frac{F_0^2}{\Delta^2}} = \frac{F_0}{\Delta}, \quad \cos\delta = \frac{A}{R} = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \text{and} \quad \sin\delta = \frac{B}{R} = \frac{\gamma\omega}{\Delta}.$$

We can write

$$R = \frac{F_0}{\Delta}, \quad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \text{and} \quad \sin\delta = \frac{\gamma\omega}{\Delta},$$

	$\cos(\omega t)$	$\sin(\omega t)$
$U(t)$	A	B
$U'(t)$	ωB	$-\omega A$
$U''(t)$	$-\omega^2 A$	$-\omega^2 B$
LHS	$(k-m\omega^2)A+\gamma\omega B$	$(k-m\omega^2)B-\gamma\omega A$
RHS	F_0	0

Table 3.1: Draft table for the derivation.

where $\Delta = \sqrt{m^2(\omega_0^2 - m\omega^2)^2 + \gamma^2\omega^2}$ and $\omega_0 = \sqrt{k/m}$. We investigate how the amplitude R of the steady-state oscillation depends on the frequency ω of the external force. By some algebraic manipulations, we obtain that

$$\frac{Rk}{F_0} = \left[\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \Gamma\left(\frac{\omega}{\omega_0}\right)^2\right]^{-1/2}, \quad \text{where } \Gamma = \frac{\gamma^2}{mk}. \tag{3.63}$$

Observe that the quantity Rk/F_0 is the ratio of the amplitude R of the forced response to F_0/k, the static displacement of the spring produced by a force F_0. For low frequency regime, that is $\omega \to 0$, it follows that the ratio $Rk/F_0 \to 1$ or $R \to F_0/k$. At the other extreme, for very high frequency regime, it implies that $R \to 0$ as $\omega \to \infty$. At an intermediate value of ω, the amplitude R may have a maximum.

Disclaimer: One may skip the following *tedious* yet *simple* derivation if you are not interested in it. Note that $k = m\omega_0^2$ and $\Delta = \sqrt{m^2(\omega_0^2 - m\omega^2)^2 + \gamma^2\omega^2}$. Using the formula $R = F_0/\Delta$, we have

$$\begin{aligned}
R^2 &= \frac{F_0^2}{\Delta^2} = \frac{F_0^2}{(k-m\omega^2)^2+\gamma^2\omega^2} \\
\Longrightarrow \frac{R^2k^2}{F_0^2} &= \frac{k^2}{(k-m\omega^2)^2+\gamma^2\omega^2} = \frac{1}{\left(1-\frac{m\omega^2}{k}\right)^2 + \frac{\gamma^2\omega^2}{k^2}} \\
&= \left[\left(1-\frac{m\omega^2}{m\omega_0^2}\right)^2 + \frac{\gamma^2\omega^2}{k\cdot m\omega_0^2}\right]^{-1} = \left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{\gamma^2}{mk}\cdot\left(\frac{\omega}{\omega_0}\right)^2\right]^{-1}
\end{aligned}$$

Thus, we have

$$\frac{Rk}{F_0} = \left[\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \Gamma\left(\frac{\omega}{\omega_0}\right)^2\right]^{-1/2}.$$

To find this maximum point of R, we can differentiate R with respect to ω and set the result of ω equal to zero. In this way, we find that the maximum amplitude occurs when $\omega = \omega_{\max}$ where

$$\omega_{\max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2\left(1 - \frac{\gamma^2}{2mk}\right). \tag{3.64}$$

Note that $\omega_{\max} < \omega_0$ and that $\omega_{\max}$ is close to ω_0 when γ is small. The maximum value of the amplitude R is

$$R_{\max} = \frac{F_0}{\gamma\omega_0}\left(1 - \frac{\gamma^2}{4mk}\right)^{-1/2} \approx \frac{F_0}{\gamma\omega_0}\left(1 + \frac{\gamma^2}{8mk}\right). \tag{3.65}$$

The last expression is an approximation that is only valid when $\gamma^2/(4mk)$ is very small.

Disclaimer: One may skip the following *tedious* yet *simple* derivation if you are not interested in it. Note that $R = F_0/\Delta$ with $\Delta = \sqrt{m^2(\omega_0^2 - m\omega^2)^2 + \gamma^2\omega^2}$. We can view R as a function of ω such that

$$R(\omega) = F_0 \left(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right)^{-1/2}.$$

Differentiating R with respect to ω, we obtain that

$$\begin{aligned}\frac{dR}{d\omega} &= -\frac{F_0}{2}\left(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right)^{-3/2}\left[-2m^2(\omega_0^2 - \omega^2)(2\omega) + 2\gamma^2\omega\right] \\ &= -\frac{F_0}{2}\frac{-4m^2(\omega_0^2 - \omega^2)\omega + 2\gamma^2\omega}{\left(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right)^{3/2}}.\end{aligned}$$

If we set $R'(\omega) = 0$, then we should have

$$-2m^2(\omega_0^2 - \omega^2)\omega + \gamma^2\omega = -\omega(2m^2\omega^2 - (2m^2\omega_0^2 - \gamma^2))0.$$

Solving the above equation for ω (usually we assume that $\omega \neq 0$ since this is the frequency of the external force), we obtain that

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{2m^2}.$$

We denote $\omega_{\max}$ to be the ω in the above equation. Substituting this $\omega_{\max}$ back to the amplitude function $R(\omega)$, we have

$$\begin{aligned}R_{\max} = R(\omega_{\max}) &= F_0\left[m^2\left(\omega_0^2 - \omega_0^2 + \frac{\gamma^2}{2m^2}\right)^2 + \gamma^2\omega_0^2 - \frac{\gamma^4}{2m^2}\right]^{-1/2} \\ &= F_0\left[\frac{\gamma^4}{4m^2} + \gamma^2\omega_0^2 - \frac{\gamma^4}{2m^2}\right]^{-1/2} \\ &= F_0\left[\gamma^2\omega_0^2 - \frac{\gamma^4}{4m^2}\right]^{-1/2} \\ &= \frac{F_0}{\gamma\omega_0}\left(1 - \frac{\gamma^2}{4m^2\omega_0^2}\right)^{-1/2} = \frac{F_0}{\gamma\omega_0}\left(1 - \frac{\gamma^2}{4mk}\right)^{-1/2},\end{aligned}$$

using the fact that $k = m\omega_0^2$.

If the damping coefficient γ satisfies $\gamma^2 > 2mk$, then $\omega_{\max}$ given in (3.64) is imaginary; in this case, we rewrite $R'(\omega)$

$$R'(\omega) = -\frac{2m^2F_0\omega}{\Delta^3}\left(\omega^2 + \frac{\gamma^2 - 2mk}{2m^2}\right) < 0$$

since ω, F_0, m, γ, and Δ are all positive. Thus, the maximum value of R occurs at $\omega = 0$ and R is a monotone decreasing function of ω. Recall that critical damping occurs when $\gamma^2 = 4mk$.

Recall that

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}.$$

For small γ it follows from (3.65) that $R_{\max} \approx F_0/(\gamma\omega_0)$ when ω is very close to ω_0. For lightly damped systems, the amplitude R of the forced response when ω is near ω_0 is quite large even for relatively small external forces, and the smaller the value of γ, the more pronounced is this effect. This phenomenon is known as *resonance*, and it is often an important design consideration. Resonance can be either good or bad, depending on the circumstances. It must be taken very seriously in the design of structures, such as buildings and bridges, where it can produce instabilities that might lead to the catastrophic failure of the structure. On the other hand, resonance can be put to good use in the design of instruments, such as seismographs, that are intended to detect weak periodic incoming signals.

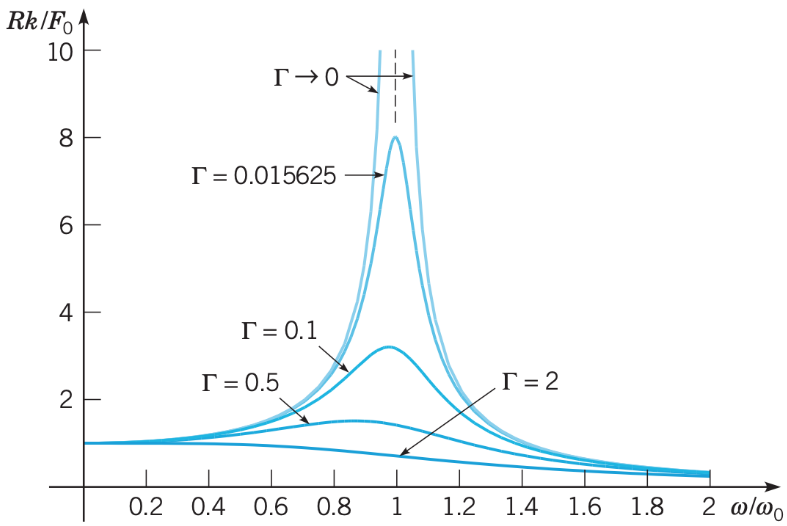


Figure 3.4: Periodic forced vibration with damping.

Figure 3.4 shows some representative curves of Rk/F_0 against ω/ω_0 for several values of $\Gamma = \gamma^2/(mk)$. We refer to Γ as a damping parameter. It is interesting to look at the limiting case when $\Gamma \to 0$. Recall that

$$\frac{Rk}{F_0} = \left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \Gamma\left(\frac{\omega}{\omega_0}\right)^2\right]^{-1/2}, \quad \text{where } \Gamma = \frac{\gamma^2}{mk}.$$

It follows that

$$\frac{Rk}{F_0} = \left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^{-1} \to +\infty \quad \text{as } \frac{\omega}{\omega_0} \to 1.$$

Hence, the variable Rk/F_0 is asymptotic to the vertical line $\omega/\omega_0 = 1$, as shown in Figure 3.4. As the damping in the system increases, the peak response gradually diminishes.

Figure 3.4 also demonstrates the usefulness of dimensionless variables (i.e., quantities with no units). One can easily verify that each of the quantities Rk/F_0, ω/ω_0, and Γ has no unit. The importance of this observation is that the number of significant parameters (m, γ, k, F_0, and ω) in the problem has been reduced to three rather than five that appear in the original model problem. Thus, only one family of curves shown in Figure 3.4 is needed to describe the response-versus-frequency behavior of all systems governed by the spring-mass model.

Example 3.8.3 (Forced vibration without damping)**.** We now assume no damping in the spring-mass and consider the external force $F(t) = F_0\cos(\omega t)$. The spring-mass equation becomes

$$mu''(t) + ku(t) = F_0\cos(\omega t). \tag{3.66}$$

The form of the general solution of (3.66) is different, depending on whether the forcing frequency ω is different from or equal to the natural frequency $\omega_0 = \sqrt{k/m}$ of the unforced system.

1. First consider $\omega \neq \omega_0$. In this case, the general solution of (3.66) is

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t). \tag{3.67}$$

The constants c_1 and c_2 are determined by the initial conditions. The resulting motion is the sum of two periodic motions of different frequencies (ω_0 and ω) and different amplitudes as well.

In a particular case when $u(0) = 0$ and $u'(0) = 0$ (the mass is initially at rest). Then, the energy driving the system comes entirely from the external force, with no contribution from the initial conditions. In this case, the constants c_1 and c_2 are

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad c_2 = 0.$$

The solution is

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left(\cos(\omega t) - \cos(\omega_0 t)\right).$$

This is the sum of two periodic functions of different periods but the same amplitude. Making use of the trigonometric identities, we can rewrite the solution as follows:

$$u(t) = \frac{2F_0}{m} \left(\omega_0^2 - \omega^2\right) \sin\left(\frac{1}{2}(\omega_0 - \omega)t\right) \sin\left(\frac{1}{2}(\omega_0 + \omega)t\right).$$

In the regime when $|\omega_0 - \omega|$ is small, the frequency $\omega_0 + \omega$ is much greater than $|\omega_0 - \omega|$ and thus the term

$$\sin\left(\frac{1}{2}(\omega_0 + \omega)t\right)$$

is a rapidly oscillating function compared to the other sine term in the solution. Thus, the motion if a rapid oscillation with frequency $(\omega_0 + \omega)/2$ but with a slowly varying sinusoidal amplitude

$$\frac{2F_0}{m} \left|\omega_0^2 - \omega\right| \left|\sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)\right|.$$

This type of motion possessing a periodic variation of amplitude, exhibits what is called a *beat*. For example, such a phenomenon occurs in acoustics when two tuning forks of nearly equal frequency are excited simultaneously. In this case, the periodic variation of amplitude is quite apparent to the unaided ear. In electronics, the variation of the amplitude with time is called *amplitude modulation*.

2. Let us return to (3.66) and consider the case when $\omega = \omega_0$ (also refer to as *resonance*). In this case, the frequency of the forcing function is the same as the natural frequency of the system. In this case, the nonhomogeneous term $F_0 \cos(\omega t)$ is a solution to the associated homogeneous equation. In this case, the general solution to (3.66) is

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$$

Consider the case when $c_1 = c_2 = 0$ (it occurs when $u(0) = 0$ and $u'(0) = 0$). The solution to the spring-mass system is

$$u(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$$

	$\cos(\omega_0 t)$	$\sin(\omega_0 t)$	$t\cos(\omega_0 t)$	$t\sin(\omega_0 t)$
$U(t)$	0	0	A	B
$U'(t)$	A	B	$\omega_0 B$	$-\omega_0 A$
$U''(t)$	$2\omega_0 B$	$-2\omega_0 A$	$-\omega_0^2 A$	$-\omega_0^2 B$
LHS	$2m\omega_0 B$	$-2m\omega_0 A$	$(k - m\omega_0^2)A = 0$	$(k - m\omega_0^2)B = 0$
RHS	F_0	0	0	0

Table 3.2: Draft table for the derivation.

with amplitude as large as t. It becomes unbounded as $t \to +\infty$. In reality, unbounded oscillations do not occur, because the spring cannot stretch infinitely far. Moreover, as soon as u becomes large, the spring-mass model is no longer valid, since the assumption that the spring force depends linearly on the displacement requires that u be small. As we have seen, if damping is included in the model. the predicted motion remains bounded; however, the response to the input function $F_0\cos(\omega t)$ may be quite large if the damping is small and ω is close to ω_0.

Disclaimer: One may skip the following *tedious* yet *simple* derivation if you are not interested in it. Here we derive the particular solution to the equation $mu'' + ku = F_0\cos(\omega_0 t)$. Note that any linear combination of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ satisfies the associated homogeneous equation $mu'' + ku = 0$ in this case. Hence, we assume the particular solution $U(t)$ has the form

$$U(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t).$$

Using the table form (see Table 3.2), we obtain

$$2m\omega_0 B = F_0, \quad -2m\omega_0 A = 0 \implies A = 0, \quad B = \frac{F_0}{2m\omega_0}.$$

Hence, the particular solution is

$$U(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t).$$

3.9 Exercises

1. Solve the following given initial-value problems:

   - $y'' + y' - 2y = 0$, $y(0) = 1$, $y'(0) = 1$.
   - $4y'' - y = 0$, $y(-2) = 1$, $y'(-2) = -1$.
   - $3y'' - y' + 2y = 0$, $y(0) = 2$, $y'(0) = 0$.
   - $y'' - 2y' + 5y = 0$, $y(\pi/2) = 0$, $y'(\pi/2) = 2$.
   - $4y'' + 4y' + y = 0$, $y(0) = 1$, $y'(0) = 2$.
   - $9y'' - 12y' + 4y = 0$, $y(0) = 2$, $y'(0) = -1$.

2. Find the Wronskian $W[y_1, y_2; t]$ of the given pair of functions: $y_1(t) = \cos^2 t$, $y_2(t) = 1 + \cos(2t)$.

3. Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the differential equation

   $$t^2 y'' - 2y = 0 \quad t > 0.$$

   Compute also the Wronskian of y_1 and y_2 at any point t.

4. Use the method of reduction of order to find a second solution $y_2(t)$ of the given differential equation:

   $$t^2 y'' - 4ty' + 6y = 0, \quad y_1(t) = t^2.$$

   Hint: assume $y_2(t) = v(t)y_1(t)$ and substitute y_2 into the equation, then figure out $v(t)$.

5. Using the method of undetermined coefficients, find a particular solution $Y(t)$ of the following differential equations:

   - $y'' - 2y' - 3y = 3e^{2t}$.
   - $y'' + y' + 4y = e^t - e^{-t}$.
   - $y'' + 2y' = 3 + 4\sin(2t)$. **Hint:** for $y'' + 2y' = 3$, assume the particular solution is of the form $Y(t) = At$.
   - $y'' + y = t\cos(2t)$. **Hint:** assume $Y(t) = At\cos(2t) + Bt\sin(2t) + C\cos(2t) + D\sin(2t)$ with A, B, C, D to be determined.

6. Using the method of variation of parameters, find the general solution of the following differential equations:

   - $y'' + y = \tan(t)$ $(0 < t < \pi/2)$.
   - $y'' + 9y = 9\sec^2(3t)$ $(0 < t < \pi/6)$.
   - $y'' + 4y' + 4y = t^{-2}e^{-2t}$.

7. Consider a spring-mass system without damping and external force. The spring constant is $k = 60$ lb/ft (i.e., a mass weighting 10 lb stretches a spring 1/6 ft) and the mass is $m = 10/32$ lb $\cdot$ s^2/ft. Initially the mass is displaced an additional 1/6 ft and is then set in motion with initial upward velocity of 1 ft/s (i.e. $u(0) = 1/6$ ft and $u'(0) = -1$ ft/s).

   (a) Write down the differential equation describing the spring-mass system without damping. Denote $u(t)$ the position of the mass at any later time.

   (b) Solve the system written in (a) with the given initial conditions.

(c) What is the natural frequency $\omega_0 = \sqrt{k/m}$? What is the unit of the natural frequency? Determine the period T_d of the system.

(d) If we write the solution $u(t) = R\cos(\omega_0 t - \delta)$, determine the amplitude R and the phase δ of the motion (express δ in radian).

When reporting the quantities, don't forget to include their units.

8. Let $F_1(t)$ and $F_2(t)$ be two different external forces such that

$$F_1(t) = \begin{cases} F_0 \sin\left(\frac{t}{2}\right) & 0 \leqslant t \leqslant 2\pi, \\ 0 & t > 2\pi, \end{cases} \quad \text{and} \quad F_2(t) = \begin{cases} F_0 t & 0 \leqslant t \leqslant \pi, \\ F_0(2\pi - t) & \pi < t \leqslant 2\pi, \\ 0 & t > 2\pi. \end{cases}$$

Here, F_0 is a positive constant.

(a) Let u_1 be the solution to the initial-value problem

$$u'' + u = F_1(t), \quad u(0) = 0, \quad u'(0) = 0.$$

Find the formula of u_1.

(b) Let u_2 be the solution to the initial-value problem

$$u'' + u = F_2(t), \quad u(0) = 0, \quad u'(0) = 0.$$

Find the formula of u_2.

(c) Plot the solution curves for u_1 and u_2 in the same frame.

Hint: treat each time interval separately, and match the solutions in the different intervals by requiring u and u' to be continuous functions of t.

Exercise

There are 4 questions in this assignment. Answer all. Please write down your name and UIN. The deadline is **11:59 pm (CDT), Oct 21 2022**. **Remark:** Q1 is for Section 3.5. Although it is already covered in Exam 1, the Final Exam is cumulative. Q4 is optional with 5 bonus points if solved fully corrected.

1. Using the method of undetermined coefficients, find a particular solution $Y(t)$ of the following differential equations:
   - $y'' + 2y' = 3 + 4\sin(2t)$. **Hint:** for $y'' + 2y' = 3$, assume the particular solution is of the form $Y(t) = At$.
   - $y'' + y = t\cos(2t)$. **Hint:** assume $Y(t) = At\cos(2t) + Bt\sin(2t) + C\cos(2t) + D\sin(2t)$ with A, B, C, D to be determined.

2. Using the method of variation of parameters, find the general solution of the following differential equations:
   - $y'' + 9y = 9\sec^2(3t)$ $(0 < t < \pi/6)$.
   - $y'' + 4y' + 4y = t^{-2}e^{-2t}$.

3. Consider a spring-mass system without damping and external force. The spring constant is $k = 60$ lb/ft (i.e., a mass weighting 10 lb stretches a spring 1/6 ft) and the mass is $m = 10/32\ \text{lb}\cdot\text{s}^2/\text{ft}$. Initially the mass is displaced an additional 1/6 ft and is then set in motion with initial upward velocity of 1 ft/s (i.e. $u(0) = 1/6$ ft and $u'(0) = -1$ ft/s).
   (a) Write down the differential equation describing the spring-mass system without damping. Denote $u(t)$ the position of the mass at any later time.
   (b) Solve the system written in (a) with the given initial conditions.
   (c) What is the natural frequency $\omega_0 = \sqrt{k/m}$? What is the unit of the natural frequency? Determine the period T of the system.
   (d) If we write the solution $u(t) = R\cos(\omega_0 t - \delta)$, determine the amplitude R and the phase δ of the motion (express δ in radian).

   Remark: When reporting the quantities, don't forget to include their units.

4. (Optional, 5 bonus points) This problem studies the spring-mass model without damping term but external force. The external force only lasts for finite time from 0 to 2π. Let $F_1(t)$ and $F_2(t)$ be two different external forces such that

$$F_1(t) = \begin{cases} F_0 \sin\left(\frac{t}{2}\right) & 0 \leqslant t \leqslant 2\pi, \\ 0 & t > 2\pi, \end{cases} \quad \text{and} \quad F_2(t) = \begin{cases} F_0 t & 0 \leqslant t \leqslant \pi, \\ F_0(2\pi - t) & \pi < t \leqslant 2\pi, \\ 0 & t > 2\pi. \end{cases}$$

Here, F_0 is a positive constant.

(a) Let u_1 be the solution to the initial-value problem

$$u'' + u = F_1(t), \quad u(0) = 0, \quad u'(0) = 0.$$

Find the formula of u_1.

(b) Let u_2 be the solution to the initial-value problem

$$u'' + u = F_2(t), \quad u(0) = 0, \quad u'(0) = 0.$$

Find the formula of u_2.

(c) Plot the solution curves for u_1 and u_2 in the same frame.

Hint: treat each time interval separately, and match the solutions in the different intervals by requiring u and u' to be continuous functions of t.

Reference Solutions

1. (a) Let y_1 be a particular solution to $y'' + 2y' = 3$ and y_2 be a particular solution to $y'' + 2y' = 4\sin(2t)$, respectively. The desired $Y(t)$ is the sum of y_1 and y_2: i.e., $Y(t) = y_1(t) + y_2(t)$. Assume that

$$y_1(t) = At + Bt^2 \implies y_1'(t) = A + 2Bt, \quad y_1''(t) = 2B.$$

Hence, we have

$$y_1'' + 2y_1' = (2B + 2A) + 4Bt = 3 \implies 4B = 0, \quad 2(B + A) = 3.$$

Hence, we have $A = 3/2$ and $B = 0$. We have $y_1(t) = 3t/2$. For y_2, we assume that

$$y_2(t) = C\cos(2t) + D\sin(2t) \implies y_1'(t) = 2D\cos(2t) - 2C\sin(2t), \quad y_2''(t) = -4C\cos(2t) - 4D\sin(2t).$$

Substituting these expression back to the equation, we have

$$(-4C + 4D)\cos(2t) + (-4C - 4D)\sin(2t) = 4\sin(2t) \implies C - D = 0, \quad C + D = -4.$$

Thus, we have $C = -2$ and $D = -2$. That is, we have $y_2(t) = -2\cos(2t) - 2\sin(2t)$. Consequently, we have

$$Y(t) = y_1(t) + y_2(t) = \frac{3}{2}t - 2\cos(2t) - 2\sin(2t).$$

(b) For $y'' + y = t\cos(2t)$, we assume that $Y(t) = At\cos(2t) + Bt\sin(2t) + C\cos(2t) + D\sin(2t)$. See Table 3.3 for the derivation.

	$t\cos(2t)$	$t\sin(2t)$	$\cos(2t)$	$\sin(2t)$
$Y(t)$	A	B	C	D
$Y'(t)$	$2B$	$-2A$	$A + 2D$	$B - 2C$
$Y''(t)$	$-4A$	$-4B$	$4B - 4C$	$-4A - 4D$
LHS $= Y'' + Y$	$-3A$	$-3B$	$4B - 3C$	$-4A - 3D$
RHS	1	0	0	0

Table 3.3: Draft table for the derivation.

Hence, we have

$$-3A = 1, \quad -3B = 0, \quad 4B - 3C = 0, \quad -4A - 3D = 0$$

and therefore, $A = -1/3$, $B = C = 0$, and $D = 4/9$. The particular solution $Y(t)$ reads

$$Y(t) = -\frac{1}{3}t\cos(2t) + \frac{4}{9}\sin(2t).$$

2. (a) One can show that $y_1(t) = \cos(3t)$ and $y_2(t) = \sin(3t)$ form a fundamental set of solutions to the homogeneous equation $y'' + 9y = 0$. Thus, we can assume that the particular solution of $y'' + 9y = 9\sec^2(3t)$ has the form

$$y(t) = u_1(t)\cos(3t) + u_2(t)\sin(3t).$$

Hence, u_1' and u_2' satisfy

$$\begin{aligned} u_1'(t)\cos(3t) + u_2'(t)\sin(3t) &= 0, \\ -3u_1'(t)\sin(3t) + 3u_2'(t)\cos(3t) &= 9\sec^3(3t). \end{aligned}$$

Therefore, we have

$$u_1'(t) = -3\frac{\sin(3t)}{\cos^2(3t)} \quad \text{and} \quad u_2'(t) = \frac{3}{\cos(3t)}.$$

Integrating u_1' and u_2' respect to t, we have

$$u_1(t) = -3\int \frac{\sin(3t)}{\cos^2(3t)}\,dt = \int \frac{d\cos(3t)}{\cos^2(3t)} = -\frac{1}{\cos(3t)} + c_1,$$

$$u_2(t) = 3\int \frac{dt}{\cos(3t)} = \log\left(\frac{\sin(1.5t)+\cos(1.5t)}{\cos(1.5t)-\sin(1.5t)}\right) + c_2.$$

Hence, the general solution is

$$y(t) = c_1\cos(3t) + c_2\sin(3t) - 1 + \sin(3t)\log\left(\frac{\sin(1.5t)+\cos(1.5t)}{\cos(1.5t)-\sin(1.5t)}\right).$$

(a) One can show that $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions to the homogeneous equation $y'' + 4y' + 4y = 0$. Thus, we can assume that the particular solution of $y'' + 4y' + 4y = t^{-2}e^{-2t}$ has the form

$$y(t) = u_1(t)e^{-2t} + u_2(t)te^{-2t}.$$

Hence, u_1' and u_2' satisfy

$$\begin{aligned} u_1'(t) + tu_2'(t) &= 0, \\ -2u_1'(t) + (1-2t)u_2'(t) &= t^{-2}. \end{aligned}$$

Therefore, we have

$$u_1'(t) = -\frac{1}{t} \quad \text{and} \quad u_2'(t) = \frac{1}{t^2}.$$

Integrating u_1' and u_2' respect to t, we have

$$u_1(t) = -\log(t) + c_1,$$

$$u_2(t) = -\frac{1}{t} + c_2.$$

Hence, the general solution is

$$y(t) = c_1e^{-2t} + c_2te^{-2t} - \log(t)e^{-2t} - e^{-2t} = \tilde{c}_1e^{-2t} + c_2te^{-2t} - \log(t)e^{-2t}.$$

3. (a) The differential equation is $mu''(t) + ku(t) = 0$ where $m = 5/16$ lb $\cdot$ s^2/ft and $k = 60$ lb/ft.
(b) The initial condition is $u(0) = 1/6$ ft and $u'(0) = -1$ ft/s. Hence, the system is

$$\frac{5}{16}u''(t) + 60u(t) = 0, \quad u(0) = \frac{1}{6}, \quad u'(0) = -1$$

with appropriate units. Hence, we have

$$u(t) = A\cos\left(8\sqrt{3}t\right) + B\sin\left(8\sqrt{3}t\right).$$

Using the initial conditions, we have $A = 1/6$ and $B = -1$. Hence, we have

$$u(t) = \frac{1}{6}\cos\left(8\sqrt{3}t\right) - \sin\left(8\sqrt{3}t\right).$$

(c) The natural frequency $\omega_0 = \sqrt{k/m} = 8\sqrt{3}$ has the unit ft/s. The period of the system is $T = \sqrt{3}\pi/12$ s.
(d) Finally, the amplitude R satisfies

$$R = \sqrt{A^2 + B^2} = \frac{\sqrt{37}}{6} \text{ ft}$$

and the phase δ satisfies

$$\tan\delta = \frac{B}{A} = -6 \implies \delta \approx -1.40565$$

in radian.

3.10 Suggested Practice Problems

1. The differential equation (with solution $y = y(x)$)

$$y'' + \delta(xy' + y) = 0,$$

where δ is a constant, arises in the study of turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution, and then find the general solution in the form of an integral.

Solution. By direct computation, we have

$$y_1'(x) = -\delta x y_1(x), \quad y_1''(x) = -\delta y_1 + \delta^2 x^2 y_1(x).$$

Then, we have

$$y_1'' + \delta(xy_1' + y) = -\delta y_1 + \delta^2 x^2 y_1 - \delta^2 x^2 y_1 + \delta y_1 = 0.$$

We showed that y_1 did solve the equation. We use method of reduction of order to figure out $y_2 = vy_1$. Substituting $y_2 = vy_1$ we have

$$v''y_1 + 2v'y_1' + vy_1'' + \delta x(v'y_1 + vy_1') + \delta v y_1 = 0$$
$$\implies v''y_1 + (2y_1' + \delta x y_1)v' = 0$$
$$\implies v''y_1 - \delta x y_1 v' = 0 \implies v'' - \delta x v' = 0.$$

If we set $w = v'$, then we have

$$w' = \delta x w \implies w(x) = C_1 \exp(\delta x^2/2) \implies v(x) = C_1 \int_{t_0}^{x} \exp\left(\frac{\delta t^2}{2}\right) dt + C_2,$$

where C_1 and C_2 are arbitrary constants. Hence, the general solution of the equation is

$$y(x) = C_1 \int_{t_0}^{x} \exp\left(\frac{\delta(t^2 - x^2)}{2}\right) dt + C_2 \exp\left(-\frac{\delta x^2}{2}\right).$$

□

2. An equation of the form

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0, \quad t > 0 \tag{3.68}$$

where α and β are real constants, is called an **Euler equation**.

(a) Let $x = \ln t$ and calculate $y'(t)$ and $y''(t)$ in terms of dy/dx and d^2y/dx^2.

(b) Use the result of part (a) to transform (3.68) into

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y(x) = 0. \tag{3.69}$$

Observe that (3.69) has constant coefficients. If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of (3.69), then $y_1(\ln t)$ and $y_2(\ln t)$ form a fundamental set of solutions of (3.68).

(c) Let $\alpha = \beta = 1$. Find the general solution of the corresponding Euler equation.

Solution. We write $x = \ln t$ and we obtain $t = e^x$ and $dt/dx = e^x$.

(a) Using the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^x\frac{dy}{dt} = t\frac{dy}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right)t = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}.$$

(b) Using the results of part (a), the equation (3.68) becomes

$$\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + \alpha\frac{dy}{dx} + \beta y(x) = 0.$$

That is the equation (3.69), a constant coefficients second order differential equation.

(c) Let $\alpha = \beta = 1$. The equation (3.69) becomes

$$\frac{d^2y}{dx^2} + y(x) = 0 \implies y(x) = c_1\cos(x) + c_2\sin(x).$$

Therefore, the general solution of the Euler equation (in terms of t) is

$$y(t) = c_1\cos(\ln t) + c_2\sin(\ln t).$$

□

3. Consider linear second order equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0. \tag{3.70}$$

It can be put in a more suitable form for finding a solution by making a change of the independent variable. We explore these ideas in this problem. In particular, we determine conditions under which the equation (3.70) can be transformed into a differential equation with constant coefficients. Let $x = u(t)$ be the new independent variable, with the relation between x and t be specified later.

(a) Show that

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}, \quad \frac{d^2y}{dt^2} = \left(\frac{dx}{dt}\right)^2\frac{d^2y}{dx^2} + \frac{d^2x}{dt^2}\frac{dy}{dx}.$$

(b) Show that the equation (3.70) becomes

$$\left(\frac{dx}{dt}\right)^2\frac{d^2y}{dx^2} + \left(\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right)\frac{dy}{dx} + q(t)y = 0. \tag{3.71}$$

Solution. □

Chapter 5

Series Solutions of Second-Order Linear Equations

To deal with the much larger class of equations that have variable coefficients, it is necessary to extend our search for solutions beyond the familiar elementary functions of calculus. The principal tool that we need is the representation of a given function by a power series. The basic idea is similar to that in the method of undetermined coefficients: we assume that the solutions of a given differential equation have power series expansions, and then we attempt to determine the coefficients so as to satisfy the differential equation.

5.1 Review of Power Series

Keywords: convergence, absolute convergence, ratio test, radius of convergence, conditional convergence,

In this section, we briefly review the pertinent results about power series that we need.

1. A power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to *converge* at a point x if the limit

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n(x-x_0)^n$$

exists for that x. The series certainly converges for $x = x_0$; it may converge for all x, or it may converge for some values of x and not for others.

2. The power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to *converge absolutely* at a point x if the associated power series

$$\sum_{n=0}^{\infty} |a_n(x-x_0)^n|$$

converges. It can be shown that if the power series converges absolutely, then the power series also converges; however, the converse is not necessarily true.

3. **Ratio test:** If $a_n \neq 0$ and if for a fixed value of x,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n} \right| = |x-x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x-x_0|\, L,$$

then the power series converges absolutely at that value of x if $|x - x_0| L < 1$ and diverges if $|x - x_0| L > 1$. If $|x - x_0| L = 1$, the ratio test is inconclusive.

Example 5.1.1 (Application of ratio test)**.** Consider the power series

$$\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n \quad \text{with} \quad a_n = (-1)^{n+1} n \quad \text{and} \quad x_0 = 2.$$

The ratio test gives

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = |x-2| \lim_{n\to\infty} \left| \frac{n+1}{n} \right| = |x-2|.$$

This series converges absolutely for $|x-2| < 1$ (i.e. $1 < x < 3$), and diverges for $|x-2| > 1$. The values of x corresponding to $|x-2| = 1$ are $x = 1$ and $x = 3$. For $x = 1$, the n-th term of the series reads

$$(-1)^{n+1} n (-1)^n = (-1)^{2n+1} n = -n,$$

which does not approach 0 as $n \to \infty$. Similarly when $x = 3$ the n-th term of the series does not approach 0 as $n \to \infty$. Hence, the power series diverges for $x \leqslant 1$ and $x \geqslant 3$.

4. If the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges $x = x_1$, it converges absolutely for any x satisfying $|x - x_0| < r$ for $r = |x_1 - x_0|$; and if the series diverges at $x = x_1$, it diverges for any x satisfying $|x - x_0| > r$ with $r = |x_1 - x_0|$.

5. Typically for a power series, there is a positive number ρ called *radius of convergence*, such that the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely for $|x - x_0| < \rho$ and diverges for $|x - x_0| > \rho$. The interval $(x_0 - \rho, x_0 + \rho)$ is called the *interval of convergence* of the power series. See Figure 5.1 below. It is important to note that the series may either converge or diverge when $|x - x_0| = \rho$.

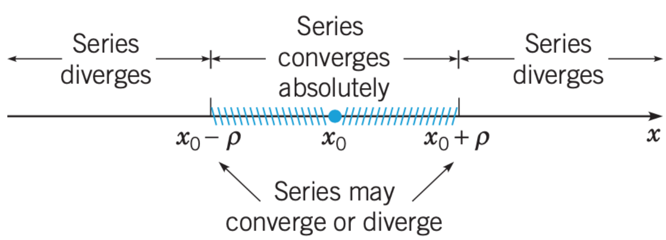


Figure 5.1: The interval of convergence of a power series.

Example 5.1.2 (Determine radius of convergence)**.** Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n 2^n}.$$

Solution. We first apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n 2^n}{(x+1)^n} \right| = \frac{|x+1|}{2} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x+1|}{2}.$$

Thus, the series converges absolutely for $|x+1| < 2$, that is for $-3 < x < 1$, and diverges for $|x+1| > 2$. The radius of convergence of the power series is $\rho = 2$. Finally, we check the end-points of the interval of convergence. At $x = 1$, the series becomes the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges. At $x = -3$, we have

$$\sum_{n=1}^{\infty} \frac{(-3+1)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Recognizing this as the alternating harmonic series, we recall that it converges but does not converge absolutely. The power series is said to converge conditionally at $x = -3$. To summarize, the given power series converges for $-3 \leqslant x < 1$ and diverges otherwise. It converges absolutely for $-3 < x < 1$ and has a radius of convergence 2. □

From now on, we assume that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ converge to $f(x)$ and $g(x)$, respectively, for $|x-x_0| < \rho$ with $\rho > 0$. Then,

6. The two series can be added or subtracted term by term. That is,

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-x_0)^n.$$

   The resulting series converges at least for $|x-x_0| < \rho$.

7. The two series can be formally multiplied, and we have

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-x_0)^n\right) = \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$

   where for any index n

$$c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0.$$

   The resulting series converges at least for $|x-x_0| < \rho$. Further if $b_0 \neq 0$, then $g(x_0) \neq 0$, and the series for $f(x)$ can be formally divided by the series for $g(x)$. That is,

$$\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} d_n(x-x_0)^n.$$

   In most cases the coefficients d_n can be most easily obtained by equating coefficients in the equivalent relation

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \left(\sum_{n=0}^{\infty} d_n(x-x_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-x_0)^n\right).$$

   In the case of division, the radius of convergence of the resulting power series may be less than the original radius of convergence ρ.

8. The limit function f of the power series is continuous and has derivatives of all orders of $|x - x_0| < \rho$. Moreover, f', f'', and any high order derivatives can be computed by differentiating the series term by term; that is,

$$f'(x) = a_1 + 2a_2(x - x_0) + \cdots + na_n(x - x_0)^{n-1} + \cdots = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}.$$

$$f''(x) = 2a_2 + 6a_3(x - x_0) + \cdots + n(n-1)a_n(x - x_0)^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2},$$

and so forth, and each of the series converges absolutely for $|x - x_0| < \rho$.

9. The series with the value of a_n given by

$$a_n = \frac{f^{(n)}(x_0)}{n!},$$

where $f^{(n)}$ is the n-th order derivative of f, is called the *Taylor series* for the function f at $x = x_0$.

10. If $\sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} b_n(x - x_0)^n$ for each x in some open interval with center x_0, then $a_n = b_n$ for all n. In particular, if $\sum_{n=0}^{\infty} a_n(x - x_0)^n = 0$ for each such x, then $a_0 = a_1 = \cdots = a_n = \cdots = 0$.

A function f that has a Taylor series expansion at $x = x_0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

with a radius of convergence $\rho > 0$, is said to be *analytic* at $x = x_0$. All of the familiar functions of calculus are analytic except perhaps at certain easily recognized points. For example, $\sin x$ and e^x are analytic everywhere, $1/x$ is analytic except at $x = 0$, and $\tan x$ is analytic except at odd multiples of $\pi/2$. According to statements 6 and 7 above, if f and g are analytic at x_0, then $f \pm g$, $f \cdot g$, and f/g (provided that $g(x_0) \neq 0$) are also analytic at $x = x_0$. In many respects the natural context for the use of power series is the complex plane. The methods and results of this chapter nearly always can be directly extended to differential equations in which the independent and dependent variables are complex-valued.

We remark that the index of summation in an infinite series is a dummy parameter just as the integration variable in a definite integral is a dummy variable. Thus, it is immaterial which letter is used for the index of summation. For example, we have

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{2^j x^j}{j!}.$$

Just as we make changes of the variable of integration in a definite integral, we find it convenient to make changes of summation indices in calculating series solutions of differential equations.

Example 5.1.3. Write $\sum_{n=2}^{\infty} a_n x^n$ as a series whose first term corresponds to $n = 0$ rather than $n = 2$.

Solution. Let $m = n - 2$; then $n = m + 2$, and $n = 2$ corresponds to $m = 0$. Hence, we have

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_{m+2} x^{m+2}.$$

Here, m is just a dummy parameter and we could write

$$\sum_{m=0}^{\infty} a_{m+2} x^{m+2} = \sum_{n=0}^{\infty} a_{n+2} x^{n+2}.$$

□

Example 5.1.4. Write $\sum_{n=2}^{\infty} (n+2)(n+1)a_n x^{n-2}$ as a series whose generic term involves $(x-x_0)^n$ rather than $(x-x_0)^{n-2}$.

Solution. Let $m = n - 2$; then $n = m + 2$, and $n = 2$ corresponds to $m = 0$. Hence, we have

$$\sum_{n=2}^{\infty} (n+2)(n+1)a_n(x-x_0)^{n-2} = \sum_{m=0}^{\infty} (m+4)(m+3)a_{m+2}(x-x_0)^m.$$

Here, m is just a dummy parameter and we could write

$$\sum_{m=0}^{\infty} (m+4)(m+3)a_{m+2}(x-x_0)^m = \sum_{n=0}^{\infty} (n+4)(n+3)a_{n+2}(x-x_0)^n.$$

□

Example 5.1.5. Assume that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

for all x. Determine what this implies about the coefficients a_n's.

Solution. By shifting the indices on the left-hand side of the above equation, we get

$$\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n$$

for all values of x. That is, we have

$$(n+1)a_{n+1} = a_n \iff a_{n+1} = \frac{a_n}{n+1}$$

for all $n \in \mathbb{N}$. That means,

$$a_1 = a_0, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2} = \frac{a_0}{2!}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{6} = \frac{a_0}{3!},$$

and in general, we have

$$a_n = \frac{a_0}{n!},$$

for all $n \in \mathbb{N}$ (recall that $0! = 1$). Hence, we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n = a_0 e^x$$

recalling that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

for all values of x. □

5.2 Series Solutions Near an Ordinary Point: Part 1

Keywords: Ordinary points, Singular points

In this section, we introduce how to use power series to solve and represent solutions to some differential equations. So far we studied methods of solving second-order linear differential equations with constant coefficients in Chapter 3. We now consider methods of solving second-order linear equations when the coefficients are functions of the independent variable, i.e.,

$$P(t)y''(t) + Q(t)y'(t) + R(t)y = 0, \tag{5.1}$$

since the procedure for the corresponding nonhomogeneous equation is similar and can be obtained via the method of variation of parameters.

In many applications, $P(t)$, $Q(t)$, and $R(t)$ are polynomials of t. However, as we will see, the method of solution is also applicable when P, Q, and R are general analytic functions. For the present, suppose that they are all polynomials and that there is no factor $(x - c)$ that is common to all three of them. If there is such a common factor $(x - c)$, then divide it out before proceeding. Suppose that we wish to solve (5.1) in the neighborhood of a point x_0. The solution in an interval containing x_0 is closely associated with the behavior of P in that interval.

A point x_0 such that $P(x_0) \neq 0$ is called an *ordinary point* of the equation. If P is continuous, it follows that there is an open interval containing x_0 in which $P(x)$ is never zero in that interval (denoted as I), and we can divide (5.1) by $P(x)$ to obtain

$$y''(t) + p(t)y'(t) + q(t)y = 0, \tag{5.2}$$

where $p(t) = Q(t)/P(t)$ and $q(t) = R(t)/P(t)$ are continuous functions on the same interval I.

On the other hand, if $P(x_0) = 0$, then x_0 is called a *singular point* of equation (5.1). In this case, since $(x - x_0)$ is not a common factor of P, Q, and R, at least one of $Q(x_0)$ and $R(x_0)$ is not zero. We discuss the case of solution near a singular point in later sections.

Instead of using the independent variable symbol t, we use x in this chapter. We now take up the problem of solving (5.1) in the neighborhood of an ordinary point x_0. This time, we look for solutions $y(x)$ of the following form

$$y(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{5.3}$$

and assume this series converges in the interval $|x - x_0| < \rho$ for some number $\rho > 0$. While at first sight it may appear unattractive to seek a solution in the form of power series, this is actually a convenient and useful form for a solution. Within the convergent intervals, power series behave very much like polynomials and are easy to manipulate both analytically and numerically. Indeed, even if we can obtain a solution in terms of elementary functions, such as exponential or trigonometric functions, we are likely to need a power series or some equivalent expression if we want to evaluate the solution numerically or to plot its graph.

To determine a_n in (5.3), simply substitute the series (5.3) and its derivatives in the equation (5.1) and equating the coefficients for both left- and right-hand sides.

Example 5.2.1. Find a series solution of the equation

$$y'' + y = 0, \quad -\infty < x < \infty.$$

Solution. As we know, the solution to this equation can be expressed as linear combination of $\sin x$ and $\cos x$. In fact, we do not need the power series method to solve it. However, this example illustrates the use of power series in a relatively simple case. In this case, $P(x) = 1 \neq 0$ for all x. Hence, every point in the real axis is an ordinary point of the equation.

Let $x_0 = 0$. We look for a solution in the form of power series at $x = 0$ as follows:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

and assume the series converges in some interval $|x| < \rho$. Differentiating the above expression, we get

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

$$y''(x) = 2a_2 + 6a_3 x + \cdots = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substituting back to the equation, we get

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0.$$

For this equation to be satisfied for all x, the coefficient of each power of x must be zero; and we conclude that

$$(n+2)(n+1)a_{n+2} + a_n = 0 \quad \text{for all } n = 0, 1, 2, 3, \cdots .$$

This is a *recurrence relation* in the sense that if we know a_0 and a_1 in advance, all the coefficients can be obtained in terms of a_0 and a_1. For example, we have

$$a_2 = -\frac{a_0}{1 \cdot 2}, \quad a_4 = -\frac{a_2}{3 \cdot 4} = (-1)^2 \frac{a_0}{4!}, \quad a_6 = -\frac{a_4}{5 \cdot 6} = (-1)^3 \frac{a_1}{6!},$$

and in general, we have

$$a_{2k} = (-1)^k \frac{a_0}{(2k)!} \quad \text{for } k = 1, 2, 3, \cdots .$$

Similarly, we have

$$a_3 = -\frac{a_1}{3 \cdots 2}, \quad a_5 = -\frac{a_3}{5 \cdot 4} = (-1)^2 \frac{a_1}{5!}, \quad a_7 = -\frac{a_5}{7 \cdot 6} = (-1)^3 \frac{a_0}{7!},$$

and in general, we have

$$a_{2k+1} = (-1)^k \frac{a_1}{(2k+1)!} \quad \text{for } k = 1, 2, 3, \cdots .$$

Then, the solution y can be represented as

$$y(x) = a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \cdots = a_0 \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \right) + a_1 \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \right).$$

We denote

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Using the ratio test, we can show that y_1 and y_2 converges for all x. For example, for y_1 we have

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(2n+2)! x^{2n+2}} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right| = \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} = 0.$$

That is, the series y_1 converges for all values of x. Similarly, y_2 converges for all values of x. Indeed, the series for $y_1(x)$ is exactly the Taylor series for $\cos x$ at $x = 0$ while y_2 is the Taylor series of $\sin x$ at $x = 0$. Hence, we recover the solution to be

$$y = a_0 \cos x + a_1 \sin x.$$

Notice that no conditions are imposed on a_0 and a_1; hence they are arbitrary. They can be determined by the initial conditions for $y(0)$ and $y'(0)$. □

Example 5.2.2. Find a series solution in powers of x of Airy's equation

$$y'' - xy = 0, \quad -\infty < x < \infty.$$

Solution. For this equation, we have $P(x) = 1$ and hence every point is an ordinary point. We assume that the solution has the power series form as follows:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and the series converges in some interval $|x| < \rho$. As an the previous example, we have

$$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

As a result, the differential equation become

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2}x^n - a_n x^{n+1}\right] = 0.$$

Rewriting the left-hand side of the equation above, we get

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2}x^n - a_{n-1}\right] x^n = 0.$$

For this equation to be satisfied for all x in some interval, the coefficients of like power of x must be zero; hence $a_2 = 0$, and we obtain the recurrence relation

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)} \quad \text{for all } n = 1, 2, 3, \cdots .$$

From the recursion above, if we are given a_0, then we know a_3, a_6, a_9, and so on. Similarly, if we know a_1, then we know a_4, a_7, and so on. Finally we have

$$a_2 = 0 \implies a_5 = a_8 = \cdots = a_{3n-3} = 0 \quad \text{for all } n = 1, 2, \cdots, .$$

Hence, we can write

$$a_3 = \frac{a_0}{3 \cdot 2}, \quad a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}, \quad a_9 = \frac{a_6}{9 \cdot 8} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9},$$

and this suggest the general formula

$$a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)} \quad \text{for any } n = 1, 2, \cdots, .$$

Similarly, we have

$$a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)} \quad \text{for any } n = 1, 2, \cdots,.$$

Therefore, the general solution of Airy's equation is

$$\begin{aligned} y(x) = a_0 &\left[1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \cdots + \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)} + \cdots \right] \\ + a_1 &\left[x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)} + \cdots \right]. \end{aligned} \tag{5.4}$$

□

We can also express the solution to the Airy's equation in terms of the power of $(x - x_0)$ with $x_0 \neq 0$.

Example 5.2.3. Find a series solution in powers of $x - 1$ of Airy's equation

$$y'' - xy = 0, \quad -\infty < x < \infty.$$

Solution. The point $x = 1$ is an ordinary point of the Airy's equation, and thus we look for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} b_n (x-1)^n,$$

where we assume that the series converges in some interval $|x - 1| < \rho$. Then,

$$y'(x) = \sum_{n=1}^{\infty} n b_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) b_{n+1} (x-1)^n,$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) b_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) b_{n+2} (x-1)^n.$$

Substituting for y and y'' in the equation, we get

$$\sum_{n=0}^{\infty} (n+2)(n+1) b_{n+2} (x-1)^n = x \sum_{n=0}^{\infty} b_n (x-1)^n = (1+x-1) \sum_{n=0}^{\infty} b_n (x-1)^n = \sum_{n=0}^{\infty} b_n (x-1)^n + \sum_{n=0}^{\infty} b_n (x-1)^{n+1}.$$

We remark that since we are expressing the solution in terms of the power of $(x - 1)$, we have to split x into the sum of 1 (zeroth power of $x - 1$) and $x - 1$. Overall, we have

$$\sum_{n=0}^{\infty} (n+2)(n+1) b_{n+2} (x-1)^n = b_0 + \sum_{n=0}^{\infty} (b_n + b_{n-1})(x-1)^n.$$

Hence, we obtain the relations

$$\begin{aligned} 2b_2 &= b_0, \\ (3 \cdot 2) b_3 &= b_0 + b_1, \\ (4 \cdot 3) b_4 &= b_1 + b_2, \\ &\vdots \\ (n+2)(n+1) a_{n+2} &= a_{n-1} + a_n. \end{aligned} \tag{5.5}$$

In general, when the above recurrence relation has more than two terms, the determination of a formula for a_n in terms of a_0 and a_1 will be fairly complicated if not impossible. □

5.3 Series Solutions Near an Ordinary Point: Part 2

Keywords: radius of convergence

We continue to study the power series methods to solve

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{5.6}$$

where P, Q, and R are polynomials, in some neighborhood of an ordinary point x_0. Assuming that (5.6) does have a solution $y = \phi(x)$ where $\phi(x)$ has a Taylor series form

$$\phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{5.7}$$

that converges for $|x - x_0| < \rho$ for some $\rho > 0$. The coefficients a_n's are determined by directly substituting the series for y in (5.6).

Consider how we might justify the statement that if x_0 is an ordinary point of (5.6), then there exist solutions of the form of Taylor series as in (5.7). We also consider the determination of the radius of convergence of such a series.

Suppose that there is a solution of (5.6) that has the form of (5.7). By differentiating (5.7) m times, and setting x equal to x_0, we get

$$m! a_m = \phi^{(m)}(x_0).$$

Hence, to compute the coefficients a_n in (5.7), we need to show that we can determine $\phi^{(n)}(x_0)$ for any number n from the differential equation (5.6). Suppose that $y = \phi(x)$ satisfies the equation (5.6) and the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Then, we have $a_0 = y_0$ and $a_1 = y_0'$. If we are solely interested in finding a solution of (5.6) without specifying any initial conditions, then a_0 and a_1 remain arbitrary. To determine $\phi^{(n)}(x_0)$ for $n \geqslant 2$ (and thus a_n), since $\phi(x)$ satisfies the equation (5.6), we have

$$\phi''(x) = -\frac{Q(x)}{P(x)}\phi'(x) - \frac{R(x)}{P(x)}\phi(x) = -p(x)\phi'(x) - q(x)\phi(x).$$

Observe that at $x = x_0$, the right-hand side of equation above is known, thus allowing us to compute $\phi''(x_0)$: setting $x = x_0$ gives

$$\phi''(x_0) = -p(x_0)a_1 - q(x_0)a_0.$$

Hence, we have

$$a_2 = \frac{\phi''(x_0)}{2!} = -\frac{1}{2}\left[p(x_0)a_1 + q(x_0)a_0\right].$$

Similarly, to obtain a_3, we have

$$3! a_3 = \phi'''(x_0) = -\left(p(x)\phi'(x) + q(x)\phi(x)\right)'\Big|_{x=x_0}.$$

That is, we have

$$a_3 = -\frac{1}{6}\left[2p(x_0)a_2 + (p'(x_0) + q(x_0))a_1 - q'(x_0)a_0\right].$$

Remark: In order to compute a_n, we need to have $(n-2)$-th order of derivatives of $p(x)$ and $q(x)$ at $x = x_0$. As a result, the existence of the power series solution to the equation (5.6) depends highly on the differentiability of p and q.

Example 5.3.1. Let $y = \phi(x)$ be a solution of the initial value problem

$$(1 + x^2)y'' + 2xy' + 4x^2y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Determine $\phi''(0)$, $\phi'''(0)$, and $\phi^{(4)}(0)$.

Solution. To find $\phi''(0)$, simply evaluate the equation when $x = 0$ and it gives

$$(1 + 0^2)\phi''(0) + 2 \cdot 0 \cdot \phi'(0) + 4 \cdot 0^2 \cdot \phi(0) = 0.$$

Hence, $\phi''(0) = 0$. To find $\phi'''(0)$, we differentiate the equation with respect to x and get

$$(1 + x^2)\phi'''(x) + 4x\phi''(x) + (2 + 4x^2)\phi'(x) + 8x\phi(x) = 0.$$

Evaluating the above equation at $x = 0$ gives

$$\phi'''(0) + 2\phi'(0) = 0 \implies \phi'''(0) = -2$$

since $\phi'(0) = y'(0) = 1$. Finally, to find $\phi^{(4)}(0)$, we differentiate one more time and gives

$$(1 + x^2)\phi^{(4)}(x) + 6x\phi'''(x) + (6 + 4x^2)\phi''(x) + 16x\phi'(x) + 8\phi(x) = 0.$$

Evaluating the above equation at $x = 0$ gives

$$\phi^{(4)}(0) + 6\phi''(0) + 8\phi(0) = 0.$$

Hence, we have $\phi^{(4)}(0) = 0$. □

One of the important aspect of the power series method is on the radius of convergence of the solution series.

1. One can show that (though not easy) the radius of convergence of the solution to the equation (5.6) is at least as large as the minimum of the radii of convergence of the series for the function p and q with $p = Q/P$ and $q = R/P$ in the equation (5.6).

2. For the fraction $p(x) = Q(x)/P(x)$, one can show that it has convergent power series expansion at $x = x_0$ if $P(x_0) \neq 0$. If $Q(x)$ and $P(x)$ have no common factor, then the radius of convergence of the power series for $p(x) = Q(x)/P(x)$ at the point x_0 is precisely the distance from x_0 to the nearest zero of $P(x)$.

3. In determining this distance, we must remember that $P(x) = 0$ may have complex roots, and these must also be considered.

Example 5.3.2. Determine the radius of convergence of the Taylor series for $\dfrac{1}{1 + x^2}$ at $x = 0$.

Solution. One can write

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Note that we have $Q(x) = 1$ and $P(x) = 1 + x^2$ in this case. The roots of $P(x) = 0$ are $\pm i$. The distance between 0 and i (or $-i$) is 1. Hence, the radius of convergence of the power series at $x = 0$ is 1. □

Example 5.3.3. Determine the radius of convergence of the Taylor series for $\dfrac{1}{x^2-2x+2}$ at $x = 1$.

Solution. One can write

$$\frac{1}{x^2-2x+2} = \frac{1}{1+(x-1)^2} = \sum_{n=0}^{\infty}(-1)^n(x-1)^{2n}.$$

Note that we have $Q(x) = 1$ and $P(x) = x^2 - 2x + 2$ in this case. The roots of $P(x) = 0$ are $1 \pm i$. The distance between 1 and $1 + i$ (or $1 - i$) is 1. Hence, the radius of convergence of the power series at $x = 1$ is 1. □

Recall that the radius of convergence of the solution series is at least as large as the radii of convergence of $Q(x)/P(x)$ and $R(x)/P(x)$ in the model problem (5.6).

Example 5.3.4. Determine a lower bound for the radius of convergence of series solutions of the differential equation

$$(1 + x^2)y'' + 2xy' + 4x^2y = 0$$

at the point $x = 0$; and at the point $x = -0.5$.

Solution. Note that $P(x) = 1 + x^2$, $Q(x) = 2x$, and $R(x) = 4x^2$ are polynomials. The radii of convergence of $Q(x)/P(x)$ and $R(x)/P(x)$ depend on the roots of $P(x) = 0$. Note that the roots of $P(x) = 0$ are $\pm i$. The distance between $x = 0$ and $\pm i$ is 1, thus the radius of convergence of the solution series at $x = 0$ is 1. While the distance between $x = -0.5$ to $\pm i$ is $\sqrt{5}/2$, then the solution at $x = -0.5$ with the form

$$\sum_{n=0}^{\infty} b_n\left(x + \frac{1}{2}\right)$$

converges at least for $\left|x + \dfrac{1}{2}\right| < \dfrac{\sqrt{5}}{2}$. □

Example 5.3.5. For the differential equation

$$y'' + (\sin x)y' + (1 + x^2)y = 0$$

determine the radius of convergence of the series solution at $x = 0$ (if any).

Solution. We have $P(x) = 1$, $Q(x) = \sin(x)$ and $R(x) = 1 + x^2$. Since $P(x) \neq 0$ for all values of x, thus there is no restriction for the radius of convergence. Hence, the series solution converges for all x. Note that when finding the series solution, we need to expand $\sin x$ in terms of the power series. For example, we can write $\sin x$ as

$$\sin x = \sum_{n=0}^{n} \frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

□

5.4 Exercises

1. Seek power series solutions of the given differential equation about the given point x_0 and find the recurrence relation that the coefficients satisfy.

   (a) $y'' - xy - y = 0$, $x_0 = 0$.

   (b) $2y'' + xy' + 3y = 0$, $x_0 = 0$.

2. Determine $y''(0)$, $y'''(0)$, and $y^{(4)}(0)$, where $y(x)$ is a solution of the given initial-value problem

$$y'' + xy' + y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

3. **(Optional, no need to hand in)** Determine a lower bound for the radius of convergence of series solutions at each given point x_0 for the given differential equation

$$(x^2 - 2x - 3)y'' + xy' + 4y = 0,$$

   where $x_0 = 4$, $x_0 = -4$, and $x_0 = 0$.

4. **(Optional, no need to hand in)** Consider the initial-value problem

$$y'' + (\sin x)y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

   Assume the solution is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

   at $x = 0$. Find the first four nonzero terms in the series.

Chapter 6

The Laplace Transform

Keywords: Laplace transform, piecewise constant function, integration by parts

In this chapter, we introduce a very useful tool for solving linear differential equations. That is called **Laplace Transform**.

6.1 Definition of the Laplace Transform

Given any function f, we define the Laplace transform of f (denoted by $\mathcal{L}\{f(t)\}$ or by $F(s)$) such that

$$\mathcal{L}\{f(t)\} = F(s) := \int_0^\infty e^{-st} f(t)dt, \tag{6.1}$$

whenever this improper integral converges. We show how to compute the Laplace transform of functions via some examples.

Example 6.1.1. Find the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ for the following functions $f(t)$:

1. $f(t) \equiv 1$ (constant function).
2. $f(t) = e^{at}$, where a is any constant.
3. $f(t) = \sin(at)$, where a is any constant.
4. $f(t) = 5e^{-2t} - 3\sin(4t)$, $t \geqslant 0$.

Solution. 1. We have

$$F(s) = \int_0^\infty e^{-st}dt = -\frac{1}{s}\int_0^\infty e^{-st}d(-st) = -\frac{1}{s}e^{-st}\Big|_0^\infty = -\frac{1}{s}(0-1) = \frac{1}{s}.$$

2. To make the integrant integrable, one has to have $s > a$. Then, we have

$$F(s) = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{-(s-a)t}dt = \frac{1}{s-a}.$$

3. We have to use integration by parts:

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st}\sin(at)dt = -\frac{1}{a}\int_0^\infty e^{-st}d\cos(at)\\
&= -\frac{1}{a}\left(e^{-st}\cos(at)\Big|_0^\infty + s\int_0^\infty e^{-st}\cos(at)dt\right)\\
&= \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos(at)dt\\
&= \frac{1}{a} - \frac{s}{a^2}\int_0^\infty e^{-st}d\sin(at)\\
&= \frac{1}{a} - \frac{s}{a^2}\left(e^{-st}\sin(at)\Big|_0^\infty + s\int_0^\infty e^{-st}\sin(at)dt\right)\\
&= \frac{1}{a} - \frac{s^2}{a^2}F(s).
\end{aligned}$$

Therefore, we have

$$\left(1+\frac{s^2}{a^2}\right)F(s) = \frac{1}{a} \implies \frac{a^2+s^2}{a^2}F(s) = \frac{1}{a} \implies F(s) = \frac{a}{s^2+a^2}.$$

4. Let $f(t) = g(t) + h(t)$ with $g(t) = 5e^{-2t}$ and $h(t) = -3\sin(4t)$. Obviously, we have

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}f(t)dt = \int_0^\infty e^{-st}(g(t)+h(t))dt\\
&= \int_0^\infty [e^{-st}g(t) + e^{-st}h(t)]dt = \int_0^\infty e^{-st}g(t)dt + \int_0^\infty e^{-st}h(t)dt\\
&= \mathcal{L}\{g(t)\} + \mathcal{L}\{h(t)\}.
\end{aligned}$$

Using the results in 2. and 3., we have

$$\mathcal{L}\{g(t)\} = 5\int_0^\infty e^{-st}e^{-2t}dt = \frac{5}{s+2} \quad \text{(with } a = -2\text{)},$$

$$\mathcal{L}\{h(t)\} = -3\int_0^\infty e^{-st}\sin(4t)dt = -3\frac{4}{s^2+16} = -\frac{12}{s^2+16} \quad \text{(with } a = 4\text{)}.$$

Therefore, we have

$$F(s) = \mathcal{L}\{f(t)\} = \frac{5}{s+2} - \frac{12}{s^2+16}.$$

□

In the example above, for any functions f_1 and f_2, we note that the Laplace transform is a linear operator in the sense that

$$\mathcal{L}\{c_1f_1(t) + c_2f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\},$$

for any constants c_1 and c_2

Example 6.1.2. For piecewise constant function

$$f(t) := \begin{cases} 1, & 0 \leqslant t < 1,\\ k, & t = 1,\\ 0, & t > 1, \end{cases}$$

with $0 < k < 1$, compute its Laplace transform. The function $f(t)$ often represents a unit pulse in engineering contexts.

Solution. Since $f(t) \equiv 0$ when $t > 1$, then we just need to calculate the Laplace transform with $0 < t < 1$. We have

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^1 e^{-st} \cdot 1dt = -\frac{1}{s}\int_0^1 e^{-st} d(-st) = \frac{1-e^{-s}}{s}.$$

The Laplace transform does not depend on k even the function is discontinuous at $t = 1$. Even if $f(t)$ is not defined at this point, the Laplace transform of f remains the same. Thus, there are many functions, differing only in their value at a single point, that have the same Laplace transform. □

6.2 Solution of Initial-Value Problem

Keywords: solving IVPs by Laplace transform, partial fraction technique

In this section, we show how the Laplace transform can be used to solve initial-value problems for linear differential equations with constant coefficients. To this aim, we first relate the Laplace transform of $'$ to the transform of f. Suppose that f is a continuously differentiable function. Then, using integration by parts, we have

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)dt = \int_0^\infty e^{-st} df(t) \\ &= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt \\ &= -f(0) + s\int_0^\infty e^{-st} f(t)dt = s\mathcal{L}\{f(t)\} - f(0).\end{aligned} \tag{6.2}$$

That is, performing the Laplace transform of the derivative of f is multiplying s with the Laplace transform of the function itself and subtracting its value at $t = 0$. Moreover, if f' is still continuously differentiable, we then have

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s\left(s\mathcal{L}\{f(t)\} - f(0)\right) - f'(0) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0). \tag{6.3}$$

Suppose that we are solving the initial-value problem

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_0'. \tag{6.4}$$

The idea in using the Laplace transform to solve (6.4) is as follows:

- Use the relations (6.2) and (6.3) to transform the initial-value problem (6.4) for an unknown function $y(t)$ in the t-domain into a simpler problem (an algebraic problem) for $Y(s) = \mathcal{L}\{y(t)\}$ in the s-domain. That is, if we write $G(s) := \mathcal{L}\{g(t)\}$, we have

$$\begin{aligned}\mathcal{L}\{ay''(t)\} + \mathcal{L}\{by'(t)\} + \mathcal{L}\{cy(t)\} &= \mathcal{L}\{g(t)\} \\ a\left(s^2Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) &= G(s)\end{aligned}$$

  or equivalently

$$\left(as^2 + bs + c\right)Y(s) = G(s) + (as+b)y_0 + ay_0'. \tag{6.5}$$

  Here, we make use of the initial conditions.

- Solve (6.5) and we have

$$Y(s) = \frac{G(s)}{as^2 + bs + c} + \frac{(as+b)y_0 + ay_0'}{as^2 + bs + c} \tag{6.6}$$

- Recover the desired solution $y(t)$ from its transform $Y(s)$. The last step is known as *inverting the transform.* Usually, we have to look up the table of Laplace transform to recover the function $y(t)$ from $Y(s)$. In this case, we denote $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Frequently, a Laplace transform $F(s)$ is expressible as a sum of several terms

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s).$$

Suppose that $f_1(t) = \mathcal{L}^{-1}\{F_1(s)\}, \cdots, f_n(t) = \mathcal{L}^{-1}\{F_n(t)\}$. Then, the function

$$f(t) = f_1(t) + \cdots + f_n(t)$$

has the Laplace transform $F(s)$. The inverse Laplace transform $\mathcal{L}^{-1}$ is also a linear operator. In many problems it is convenient to make use of the linearity property by decomposing a given transform into a sum of functions whose inverse transforms are already known or can be found in the table of Laplace transform. In these cases, **partial fraction expansions** are particularly useful for this purpose.

Example 6.2.1. Using the Laplace transform, find the solution of the differential equation

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0. \tag{6.7}$$

Solution. Although this problem can be solved using the techniques presented in Chapter 3, we present the technique of Laplace transform for this simple problem to illustrate the idea. As a remark, if $g(t) = 0$ for any t, then $G(s) = \mathcal{L}\{g(t)\} = 0$ for any $s > 0$. Denote $Y(s) = \mathcal{L}\{y(t)\}$. Using the result of (6.6), we have

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}. \tag{6.8}$$

To determine the solution $y(t)$, we find the function whose Laplace transform is $Y(s)$ given in (6.8). This can be done most easily by expanding the right-hand side of (6.8) in partial fractions. Thus, we may write

$$Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} = \frac{A(s+1) + B(s-2)}{(s-2)(s+1)},$$

where A and B are to be determined. By equating numerators of the second and the fourth terms, we obtain

$$s - 1 = a(s+1) + b(s-2) = (a+b)s + (a-2b).$$

Therefore, we must have

$$a + b = 1 \quad \text{and} \quad a - 2b = -1.$$

As a result, we have $a = 1/3$ and $b = 2/3$ and

$$Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}.$$

Recall that $\mathcal{L}\{e^{at}\} = 1/(s-a)$ for any constant a, we have

$$\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t} \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}.$$

Therefore, the solution $y(t)$ is

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

□

There is essentially a one-to-one correspondence between functions and their Laplace transforms. See Table 6.1 below.

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	s^{-1}, $(s > 0)$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, n positive integer	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$, $s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$, $s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}$, $s > \|a\|$
$\cosh(at)$	$\frac{s}{s^2-a^2}$, $s > \|a\|$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s > 0$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s > 0$
$t^n e^{at}$, n positive integer	$\frac{n!}{(s-a)^{n+1}}$, $s > 0$
$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geqslant c \end{cases}$	$\frac{e^{-cs}}{s}$, $s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$

Table 6.1: Elementary Laplace Transforms

Example 6.2.2. Find the solution of the initial-value problem

$$y'' + y = \sin(2t), \quad y(0) = 2, \quad y'(0) = 1. \tag{6.9}$$

Solution. Taking the Laplace transform of the differential equation (6.9), we obtain

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2+4}$$

where the transform of $\sin(2t)$ has been obtained from line 4 of Table 6.1. Substituting for $y(0)$ and $y'(0)$ and solving for $Y(s)$, we obtain

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}.$$

Using partial fractions, we can write $Y(s)$ in the form

$$Y(s) = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4} = \frac{(as+b)(s^2+4) + (cs+d)(s^2+1)}{(s^2+1)(s^2+4)}.$$

By expanding the numerator and equating it to the numerators, we find that

$$2s^3 + s^2 + 8s + 6 = (a+c)s^3 + (b+d)s^2 + (4a+c)s + (4b+d)$$

for all s. Then, comparing coefficients of like powers of s, we have

$$a + c = 2, \quad b + d = 1, \quad 4a + c = 8, \quad 4b + d = 6.$$

Consequently, we have $a = 2$, $c = 0$, $b = 5/3$, and $d = -2/3$. Then, we have

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}.$$

From lines 4 and 5 of the Table 6.1, the solution of the given initial-value problem is

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin(2t).$$

□

6.3 Step Functions; Translation of functions

In this section, we introduce the concept of **step function**, which is very useful in many applications. We introduce the **unit step function**, denoted by $u_c(t)$, (or **Heaviside function**) to be

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geqslant c. \end{cases} \tag{6.10}$$

Here c is a given constant. We always assume that c is non-negative since as the Laplace transform involves values of $t > 0$. The graphs of $u_c(t)$ and $1 - u_c(t)$ are plotted in Figure 6.1. The function $u_c(t)$ represents that one adds a jump with unit 1 at the time $t = c$ and keep this pulse for all the time $t > c$. If one want to reduce a unit jump at $t = c$, then one should add $-u_c(t)$ at $t = c$.

We remark that $u_0(t) = 1$ when $c = 0$. Next, we present some examples of general step functions in terms of the unit step function $u_c(t)$ for any non-negative c.

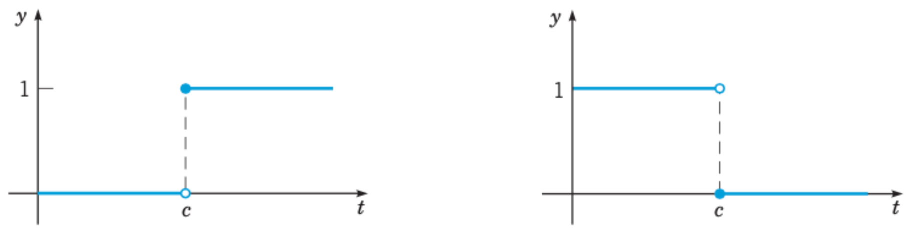


Figure 6.1: Step functions $u_c(t)$ and $1 - u_c(t)$.

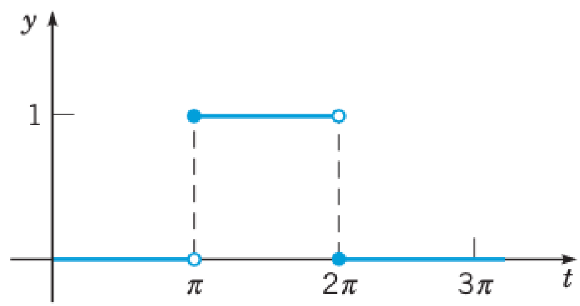


Figure 6.2: The function $h(t) = u_\pi(t) - u_{2\pi}(t)$.

Example 6.3.1. Let $h(t)$ be a function such that

$$h(t) = \begin{cases} 0, & 0 \leqslant t < \pi, \\ 1, & \pi \leqslant t < 2\pi, \\ 0, & 2\pi \leqslant t < \infty. \end{cases}$$

The function $h(t)$ can be represented by

$$h(t) = u_\pi(t) - u_{2\pi}(t)$$

for any values $t \geqslant 0$. The figure of $h(t)$ is plotted in Figure 6.2.

Example 6.3.2. Let $f(t)$ be a function such that

$$f(t) = \begin{cases} 2, & 0 \leqslant t < 4, \\ 5, & 4 \leqslant t < 7, \\ -1, & 7 \leqslant t < 9. \\ 1, & 9 \leqslant t < \infty. \end{cases}$$

Sketch the graph of $y = f(t)$ and express $f(t)$ in terms of $u_c(t)$.

Solution. The graph of $f(t)$ is plotted in 6.3. We start with the function $f_1(t) = 2$, which agrees with $f(t)$ when $0 \leqslant t < 4$. To produce the jump of three units at $t = 4$, we add $3u_4(t)$ to $f_1(t)$, obtaining

$$f_2(t) = 2 + 3u_4(t)$$

which agrees with $f(t)$ when $0 \leqslant t < 7$. The negative jump of six units at $t = 7$ corresponds to adding $-6u_7(t)$, which gives

$$f_3(t) = 2 + 3u_4 - 6u_7(t).$$

Finally, we add $2u_9(t)$ to match the jump of two units at $t = 9$. Thus, we obtain

$$f(t) = 2 + 3u_4 - 6u_7(t) + 2u_9(t).$$

□

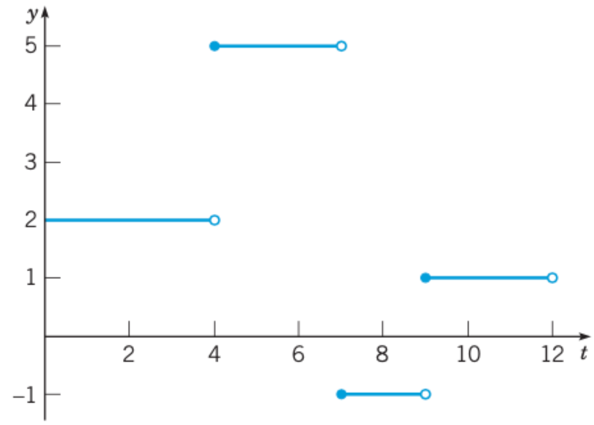


Figure 6.3: The function $f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$.

We now study the Laplace transform of the unit step function $u_c(t)$ for any non-negative constant c. We have

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t) dt = \int_c^\infty e^{-st} dt = \frac{e^{-cs}}{s} \qquad s > 0.$$

For a given function $f(t)$ defined for $t \geqslant 0$, we often want to consider the related function $g(t)$ defined by

$$g(t) = \begin{cases} 0, & t < c, \\ f(t-c), & t \geqslant c, \end{cases}$$

which represents a translation of $f(t)$ a distance c in the positive t direction and is zero for $t < c$. See Figure 6.4. Making use of the unit function, we can write $g(t)$ in terms of

$$g(t) = u_c(t) f(t-c).$$

The unit step function is particular useful in Laplace transform because of the following relation

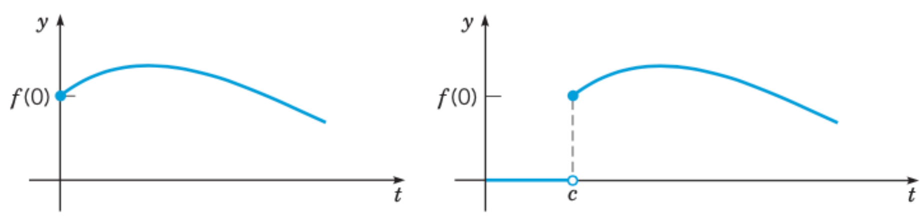


Figure 6.4: The translation of $f(t)$.

between the transform of $f(t)$ and that of its translation $u_c(t)f(t-c)$. If the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a \geqslant 0$, and if c is a positive constant, then

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs} F(s) \qquad s > a. \tag{6.11}$$

Conversely, if $f(t)$ is the inverse Laplace transform of $F(s)$, then

$$u_c(t) f(t-c) = \mathcal{L}^{-1}\{e^{-cs} F(s)\}. \tag{6.12}$$

It is obvious since

$$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c) dt = \int_c^\infty e^{-st} f(t-c) dt.$$

Introduce a new variable $\sigma = t - c$, we have

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st}e^{-s(\sigma+c)}f(\sigma)d\sigma = e^{-cs}\int_0^\infty e^{-s\sigma}f(\sigma)d\sigma = e^{-cs}F(s).$$

The equation (6.12) follows by taking the inverse transform of both sides of (6.11). We also introduce the following useful property of Laplace transforms. If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geqslant 0$ and if c is a constant, then

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c) \quad s > a + c. \tag{6.13}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct}f(t) = e^{ct}\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s-c)\}. \tag{6.14}$$

Example 6.3.3. Let $f(t)$ be given by

$$f(t) = \begin{cases} \sin(t), & 0 \leqslant t < \dfrac{\pi}{4}, \\ \sin(t) + \cos\left(t - \dfrac{\pi}{4}\right), & \dfrac{\pi}{4} \leqslant t. \end{cases}$$

It is easy to note that

$$f(t) = \sin(t) + g(t), \quad \text{where} \quad g(t) = u_{\pi/4}(t)\cos\left(t - \frac{\pi}{4}\right) = \begin{cases} 0, & 0 \leqslant t < \dfrac{\pi}{4}, \\ \cos\left(t - \dfrac{\pi}{4}\right), & \dfrac{\pi}{4} \leqslant t. \end{cases}$$

Thus, the Laplace transform of $f(t)$ becomes

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t)\} + \mathcal{L}\{g(t)\} = \mathcal{L}\{\sin(t)\} + \mathcal{L}\left\{u_{\pi/4}(t)\cos\left(t - \frac{\pi}{4}\right)\right\}.$$

Using the property (6.11), we have

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s/4}\mathcal{L}\{\cos(t)\} = \frac{1 + se^{-\pi s/4}}{s^2+1}.$$

Example 6.3.4. We can find the inverse Laplace transform of

$$F(s) = \frac{1 - e^{-2s}}{s^2}$$

using the property (6.12). Note that

$$F(s) = \frac{1}{s^2} - \frac{e^{-2s}}{s^2}.$$

Note that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \quad \text{and} \quad u_2(t)(t-2) = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$$

using the property (6.12). Therefore, we have

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = t - u_2(t)(t-2) = \begin{cases} t, & 0 \leqslant t < 2, \\ 2, & t \geqslant 2. \end{cases}$$

Example 6.3.5. Let $G(s)$ be given by

$$G(s) = \frac{1}{s^2 - 4s + 5} = \frac{1}{(s-2)^2 + 1}.$$

If we define $F(s) = (s^2 + 1)^{-1}$, then we have

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2).$$

Since $\mathcal{L}^{-1}\{F(s)\} = \sin t$, it follows from the property (6.14), we have

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{F(s-2)\} = e^{2t}\mathcal{L}^{-1}\{F(s)\} = e^{2t}\sin(t).$$

The results of this section are useful in solving differential equations, particularly those that have discontinuous forcing functions. The next section is devoted to examples illustrating this point.

6.4 Differential Equations with Discontinuous Forcing Functions

In this section, we turn our attention to some examples in which the nonhomogeneous term (**forcing function**) is discontinuous.

Example 6.4.1. Find the solution of the initial-value problem

$$2y'' + y' + 2y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \tag{6.15}$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 0, & 0 \leqslant t < 5, \\ 1, & 5 \leqslant t < 20, \\ 0, & 20 \leqslant t < \infty. \end{cases}$$

We remark that this problem governs the charge on the capacitor in a simple electric circuit with a unit voltage pulse for $5 \leqslant t < 20$. Alternatively, y may represent the response of a damped oscillator subject to the applied force $g(t)$.

Solution. The Laplace transform of (6.16) is

$$2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\} = \frac{1}{s}\left(e^{-5s} - e^{-20s}\right).$$

Making use of the initial values and solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}.$$

It is useful to write

$$Y(s) = e^{-5s}H(s) - e^{-20s}H(s) \quad \text{where} \quad H(s) = \frac{1}{s(2s^2 + s + 2)}.$$

We need to find the inverse Laplace transform $h(t) = \mathcal{L}^{-1}\{H(s)\}$ since

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{e^{-5s}H(s)\} - \mathcal{L}^{-1}\{e^{-20s}H(s)\} = u_5(t)h(t-5) - u_{20}(t)h(t-20).$$

That is, we have

$$y(t) = \begin{cases} 0 & 0 \leqslant t < 5, \\ h(t-5) & 5 \leqslant t < 20, \\ h(t-5) - h(t-20) & t \geqslant 20. \end{cases}$$

To determine $h(t)$, we use the partial fraction expansion of $H(s)$ and we have

$$H(s) = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2}.$$

Upon determining the coefficients, we find that $a = 1/2$, $b = -1$ and $c = -1/2$. Thus, we have

$$H(s) = \frac{1}{2} \cdot \frac{1}{s} - \frac{s + \frac{1}{2}}{2s^2 + s + 2} = \frac{1}{2}\left(\frac{1}{s} - \frac{s + \frac{1}{2}}{s^2 + \frac{s}{2} + 1}\right).$$

We complete the square for $s^2 + \frac{s}{2} + 1 = \left(s + \frac{1}{4}\right)^2 + \frac{15}{16} = \left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2$. We have

$$\begin{aligned} H(s) &= \frac{1}{2}\left(\frac{1}{s} - \frac{s + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} - \frac{\frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\right) \\ &= \frac{1}{2}\left(\frac{1}{s} - \frac{s + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} - \frac{1}{\sqrt{15}} \frac{\frac{\sqrt{15}}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\right) \\ &= \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \frac{s + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} - \frac{1}{2\sqrt{15}} \frac{\frac{\sqrt{15}}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}. \end{aligned}$$

By referring Table 6.1, we have

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{2} - \frac{1}{2} e^{-t/4} \cos\left(\frac{\sqrt{15}}{4} t\right) - \frac{1}{2\sqrt{15}} e^{-t/4} \sin\left(\frac{\sqrt{15}}{4} t\right).$$

□

Example 6.4.2. Find the solution of the initial-value problem

$$y'' + 4y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \tag{6.16}$$

where

$$g(t) = \begin{cases} 0, & 0 \leqslant t < 5, \\ \dfrac{t-5}{5}, & 5 \leqslant t < 10, \\ 1, & 10 \leqslant t < \infty. \end{cases}$$

This forcing function is known as **ramp loading**.

Solution. Note that

$$g(t) = \frac{1}{5}\left(u_5(t)(t-5) - u_{10}(t)(t-10)\right).$$

Then, taking the Laplace transform and use the initial conditions, we have

$$(s^2 + 4)Y(s) = \frac{e^{-5s} - e^{-10s}}{5s^2}.$$

Then, we can solve for $Y(s)$ as follows:

$$Y(s) = \frac{1}{5}(e^{-5s} - e^{-10s})H(s), \quad \text{where} \quad H(s) = \frac{1}{s^2(s^2+4)}.$$

Denote $h(t) = \mathcal{L}^{-1}\{H(s)\}$. We have

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{5}\Big(\mathcal{L}^{-1}\{e^{-5s}H(s)\} - \mathcal{L}^{-1}\{e^{-10s}H(s)\}\Big) = \frac{1}{5}\Big(u_5(t)h(t-5) - u_{10}(t)h(t-10)\Big).$$

It remains to determine $h(t)$. Using the partial fractions, we have

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4} = \frac{1}{4}\cdot\frac{1}{s^2} - \frac{1}{8}\cdot\frac{2}{s^2+2^2}.$$

It follows that

$$h(t) = \frac{t}{4} - \frac{1}{8}\sin(2t).$$

□

6.5 Impulse Functions; Dirac Delta Functions

In this section, we introduce the impulse functions, which describe the phenomena of an impulsive nature such as voltages or forces of large magnitude that act over short time intervals.

Given a function $g(t)$ such that

$$g(t) = \begin{cases} \text{very large} & t_0 - \tau < t < t_0 + \tau, \\ 0 & \text{otherwise,} \end{cases}$$

for some $\tau > 0$ and t_0 being any constant. We define the **total impulse** of $g(t)$ to be

$$I(\tau) := \int_{t_0-\tau}^{t_0+\tau} g(t)dt = \int_{-\infty}^{\infty} g(t)dt.$$

This total impulse is a measure of the strength of the forcing function.

Example 6.5.1. Let $t_0 = 0$ and τ be any positive number. Let $g(t)$ be given by

$$g(t) = d_\tau(t) := \begin{cases} \dfrac{1}{2\tau} & -\tau < t < \tau, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{6.17}$$

The total impulse $I(\tau)$ of $g(t)$ is $I(\tau) = 1$, which is independent of the value of τ, as long as $\tau \neq 0$. We let the number τ become smaller and smaller; that is, we consider the limit of $d_\tau(t)$ as $\tau \to 0^+$ keeping τ being non-negative. As a result, we obtain that

$$\lim_{\tau\to 0^+} d_\tau(t) = 0 \quad \text{for any } t \neq 0, \quad \text{but} \quad \lim_{\tau\to 0^+} d_\tau(0) = \infty. \tag{6.18}$$

In the meantime, the total impulse is still $I(\tau) = 1$ for each $\tau \neq 0$ and it follows that

$$\lim_{\tau\to 0^+} I(\tau) = 1. \tag{6.19}$$

The equations (6.18) and (6.19) define an idealized **unit impulse function** $\delta(t)$, which imparts an impulse of magnitude infinity at $t = 0$ but is zero for all values of $t \neq 0$; also it has the total impulse one. That is, the *weird function* $\delta(t)$ is defined to have the properties

$$\delta(t) = \begin{cases} \infty & t = 0, \\ 0 & t \neq 0, \end{cases} \tag{6.20}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1. \tag{6.21}$$

This function $\delta(t)$ is not an ordinary one and is known as **generalized function**. It is usually called the **Dirac delta function**. Since $\delta(t)$ corresponds to a unit impulse at $t = 0$, a unit impulse at an arbitrary point $t = t_0$ is given by $\delta(t - t_0)$. From (6.20) and (6.21), we have

$$\delta(t - t_0) = \begin{cases} \infty & t = t_0, \\ 0 & t \neq t_0, \end{cases} \tag{6.22}$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)dt = 1. \tag{6.23}$$

We define the Laplace transform of $\delta(t - t_0)$ to be

$$\mathcal{L}\{\delta(t - t_0)\} = \int_0^{\infty} e^{-st}\delta(t)dt = e^{-st_0} \tag{6.24}$$

for any $t_0 > 0$. For $t_0 = 0$, we have

$$\mathcal{L}\{\delta(t)\} = 1.$$

The Dirac delta function has the following useful property: for any function $f(t)$, it holds that

$$\int_{-\infty}^{\infty} \delta(t - t_0)f(t)dt = f(t_0)$$

for any number t_0. The following examples illustrate the use of the delta functions in solving initial-value problems with impulsive forcing functions.

Example 6.5.2. Find the solution of the initial-value problem

$$2y'' + y' + 2y = \delta(t - 5), \quad y(0) = y'(0) = 0. \tag{6.25}$$

Solution. To solve the given problem, we first take the Laplace transform of the differential equation and use the initial conditions, obtaining

$$(2s^2 + s + 2)Y(s) = e^{-5s}.$$

Thus, we have

$$Y(s) = e^{-5s} \cdot \frac{1}{2s^2 + s + 2}.$$

The right-hand side is a product of e^{-5s} with a fraction (dominator being a quadratic function $2s^2 + s + 2$). This motivates us to use

$$[Y(s) =] \, e^{-5s}\mathcal{L}\{f(t)\} = \mathcal{L}\{u_5(t)f(t - 5)\}$$

for some function f (to be determined) such that

$$\mathcal{L}\{f(t)\} = \frac{1}{2s^2 + s + 2}.$$

Since the Laplace transform of $f(t)$ is a fraction with quadratic dominator, it relates to some sine and cosine functions after completing the square. Hence, we have

$$\mathcal{L}\{f(t)\} = \frac{1}{2s^2+s+2} = \frac{1}{2}\cdot\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} = \frac{1}{2}\cdot\frac{4}{\sqrt{15}}\cdot\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2} = \frac{2}{\sqrt{15}}\cdot\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}.$$

If we denote

$$F(s) = \frac{a}{s^2+a^2} = \mathcal{L}\{\sin(at)\} \quad \text{with } a = \frac{\sqrt{15}}{4},$$

then, we have

$$\mathcal{L}\{f(t)\} = \frac{2}{\sqrt{15}}F\left(s+\frac{1}{4}\right) = \frac{2}{\sqrt{15}}\mathcal{L}\{e^{-t/4}\sin(at)\} \implies f(t) = \frac{2}{\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right).$$

Thus,

$$Y(s) = \frac{e^{-5s}}{2s^2+s+2} = \mathcal{L}\{u_5(t)f(t-5)\} \implies y(t) = u_5(t)f(t-5).$$

It is possible to write $y(t)$ in the form

$$y(t) = u_5(t)f(t-5) = \begin{cases} 0, & 0 \leqslant t < 5, \\ \frac{2}{\sqrt{15}}e^{-(t-5)/4}\sin\left(\frac{\sqrt{15}}{4}(t-5)\right), & t \geqslant 5. \end{cases}$$

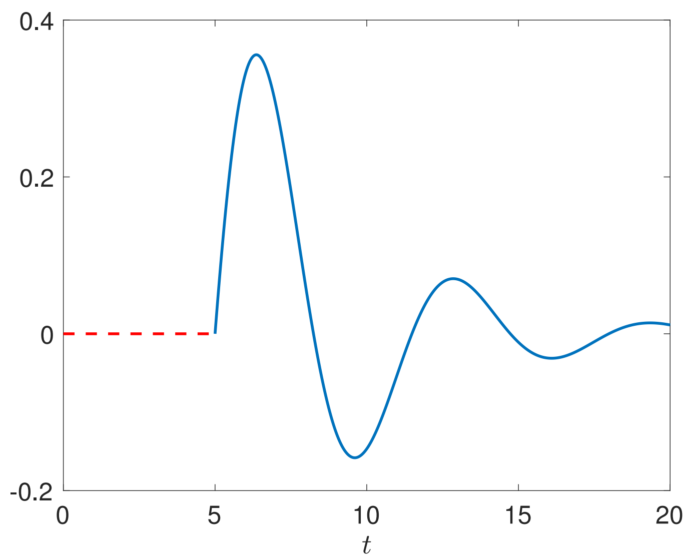


Figure 6.5: The solution $y(t)$ to the initial-value problem in Example 6.5.2.

See Figure 6.5 for the graph of the solution $y(t)$. It matches the intuition in the sense that the solution $y(t)$ is zero before the time moment $t = 5$ since the source term and the initial conditions are all zero before $t = 5$. □

Example 6.5.3. Find the solution of the initial-value problem

$$y'' + 4y = \delta(t-\pi), \quad y(0) = y'(0) = 0. \tag{6.26}$$

Solution. To solve the given problem, we first take the Laplace transform of the differential equation and use the initial conditions, obtaining

$$(s^2+4)Y(s) = e^{-\pi s} \implies Y(s) = \frac{e^{-\pi s}}{s^2+4}.$$

Note that

$$\mathcal{L}^{-1}\left\{F(s) := \frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{2}\sin(2t) =: f(t).$$

Thus, we have

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{e^{-\pi s}F(s)\} = u_\pi(t)f(t-\pi) = \frac{1}{2}u_\pi(t)\sin(2(t-\pi)) = \frac{1}{2}u_\pi(t)\sin(2t)$$

since $\sin(2t) = \sin(2t-2\pi)$. Hence, we have

$$y(t) = \frac{1}{2}u_\pi(t)\sin(2t) = \begin{cases} 0, & 0 \leqslant t < \pi, \\ \frac{1}{2}\sin(2t), & t \geqslant \pi. \end{cases}$$

□

6.6 The Convolution Integral

In this section, we introduce a special integral operation that is called **convolution**. Given any two functions f and g, we define the convolution of f and g, denoted by $f*g$, which is a function such that

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau. \tag{6.27}$$

Sometimes it is possible to identify a Laplace transform $H(s)$ as the product of two other Laplace transform $F(s)$ and $G(s)$, the latter transforms corresponding to some known functions f and g, respectively. In this even, we might anticipate that $H(s)$ is the convolution of f and g. In particular, we have the following result, referred to be the **convolution theorem**.

If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ and $H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}$, then we have

$$h(t) = (f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau = (g*f)(t). \tag{6.28}$$

According to this theorem, the transform of the convolution of two functions, rather than the transform of their ordinary product, is given by the product of the separate transforms. It is conventional to emphasize that the convolution integral can be thought of as a *generalized product* by writing

$$h(t) = (f*g)(t).$$

The operation of convolution has many of the properties of ordinary multiplication. For example, it has

- $f*g = g*f$;
- $f*(g_1+g_2) = f*g_1 + f*g_2$;

- $(f * g) * h = f * (g * h)$;
- $f * 0 = 0 * f = 0$.

Here, the zeros denote not the number zero but the function that has the value 0 for each value of t. One can easily verify those properties.

There are other properties of ordinary multiplication that the convolution integral does not have. It is not true in general that $f * 1$ is equal to f. For instance, if we take $f(t) = \cos t$, then we have

$$\begin{aligned}(f * 1)(t) &= \int_0^t f(t-\tau)\cdot 1\ d\tau = \int_0^t \cos(t-\tau)d\tau \\ &= -\sin(t-\tau)\Big|_{\tau=0}^{\tau=t} = -\sin 0 + \sin t = \sin t \\ &\neq f(t).\end{aligned}$$

Similarly, it may not be true that $f * f$ is nonnegative (e.g. when $f(t) = \sin t$). Now let us look at some examples.

Example 6.6.1. Find the inverse Laplace transform of

$$H(s) = \frac{a}{s^2(s^2+a^2)}.$$

Here, a is any real constant.

Solution. It is convenient to think of $H(s)$ as the product of

$$H(s) = F(s)G(s) \quad \text{where} \quad F(s) = \frac{1}{s^2} \quad \text{and} \quad G(s) = \frac{a}{s^2+a^2}.$$

Note that $\mathcal{L}\{t\} = F(s)$ and $\mathcal{L}\{\sin(at)\} = G(s)$. Then, using the convolution theorem, we have

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \int_0^t (t-\tau)\sin(a\tau)d\tau = \frac{at - \sin(at)}{a^2}.$$

Note that

$$\begin{aligned}\int_0^t (t-\tau)\sin(a\tau)d\tau &= t\int_0^t \sin(a\tau)d\tau - \int_0^t \tau\sin(a\tau)d\tau \\ &= \frac{t}{a}\left(-\cos(a\tau)\Big|_{\tau=0}^{\tau=t}\right) + \frac{1}{a}\int_0^t \tau d\cos(a\tau) \\ &= \frac{t}{a} - \frac{\cos(at)}{a} + \frac{1}{a}\left(\tau\cos(a\tau)\Big|_{\tau=0}^{\tau=t} - \int_0^t \cos(a\tau)d\tau\right) \\ &= \frac{t}{a} - \frac{1}{a^2}\sin(a\tau)\Big|_{\tau=0}^{\tau=t} = \frac{t}{a} - \frac{\sin(at)}{a^2} \\ &= \frac{at-\sin(at)}{a^2}.\end{aligned}$$

We remark that $h(t)$ can be found by expanding $H(s)$ in partial fractions. □

Example 6.6.2. Find the solution of the initial-value problem

$$\begin{aligned}y'' + 4y &= g(t), \\ y(0) &= 3, \\ y'(0) &= -1.\end{aligned} \tag{6.29}$$

Here, the function $g(t)$ is arbitrary. We would express the solution $y(t)$ in terms of an integral representation.

Solution. Denote $Y(s) = \mathcal{L}\{y(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. By taking the Laplace transform of the differential equation and using the initial conditions, we have

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = G(s) \implies s^2Y(s) - 3s + 1 + 4Y(s) = G(s).$$

Therefore, we have

$$(s^2+4)Y(s) = 3s - 1 + G(s) \implies Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}.$$

It is convenient to write $Y(s)$ in the form

$$Y(s) = 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}G(s).$$

The first two terms relate to the term of $3\cos(2t)$ and $\sin(2t)/2$, and the third term can be represented by the convolution of $\sin(2t)$ and $g(t)$. Hence, we have

$$y(t) = 3\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}\int_0^t \sin(2(t-\tau))g(\tau)d\tau.$$

If a specific forcing function g is given, then the integral can be evaluated. □

The convolution integral is a powerful tool for writing the solution of an initial-value problem in terms of an integral. Consider the following general (second-order) initial-value problem as follows:

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_0'. \tag{6.30}$$

Denote $Y(s) = \mathcal{L}\{y(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. Taking the Laplace transform of (6.30) and using initial conditions, we have

$$(as^2 + bs + c)Y(s) - (as+b)y_0 - ay_0' = G(s).$$

Then, we have

$$Y(s) = \frac{(as+b)y_0 + ay_0'}{as^2+bs+c} + \frac{G(s)}{as^2+bs+c}.$$

If we write

$$\Phi(s) := \frac{(as+b)y_0 + ay_0'}{as^2+bs+c} \quad \text{and} \quad \Psi(s) := \frac{G(s)}{as^2+bs+c},$$

then we have

$$Y(s) = \Phi(s) + \Psi(s)$$

and consequently

$$y(t) = \phi(t) + \psi(t)$$

with

$$\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\} \quad \text{and} \quad \psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}.$$

Now, we do some observation here. The function $\phi(t)$ is actually the solution of the homogeneous initial-value problem:

$$ay'' + by' + cy = 0, \quad y(0) = y_0, \quad y'(0) = y_0',$$

which can be obtained from (6.30) with zero forcing function. On the other hand, $\psi(t)$ is the solution of the nonhomogeneous initial-value problem

$$ay'' + by' + cy = g(t), \quad y(0) = 0, \quad y'(0) = 0,$$

in which the initial values are all zero.

Once specific values of a, b, and c are given, we can easily find $\phi(t)$ by the techniques presented in Chapter 3. To find $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$, it is convenient to write $\Psi(s)$ as

$$\Psi(s) = H(s)G(s) \quad \text{where} \quad H(s) = \frac{1}{as^2 + bs + c}.$$

The function $H(s)$ is known as the **transfer function** and depends only on the values of a, b, and c. The function $G(s)$ depends on the external force $g(t)$ that is applied to the equation. By the convolution theorem, we have

$$\psi(t) = \mathcal{L}^{-1}\{H(s)G(s)\} = \int_0^t h(t-\tau)g(\tau)d\tau,$$

where $h(t) = \mathcal{L}^{-1}\{H(s)\}$, and $g(t)$ is the given forcing function.

Referring to Example 6.6.2, we note that the transfer function is $H(s) = 1/(s^2+4)$ and $h(t) = \sin(2t)/2$. Also, the first two terms in the solution relate to the solution of the corresponding homogeneous equation that satisfies the given initial conditions.

Example 6.6.3. Solve the initial-value problem

$$y'' + y = -4\sin(2t), \quad y(0) = 0, \quad y'(0) = 2. \tag{6.31}$$

Solution. First, we split the problem into two sub-problems:

$$y'' + y = 0, \quad y(0) = 0, \quad y'(0) = 2, \tag{6.32}$$

and

$$y'' + y = -4\sin(2t), \quad y(0) = 0, \quad y'(0) = 0. \tag{6.33}$$

We write $g(t) = -4\sin(2t)$ and $G(s) = \mathcal{L}\{g(t)\}$. We denote $\phi(t)$ the solution of (6.32). One can easily have

$$\phi(t) = 2\sin(t).$$

We denote $\psi(t)$ the solution of (6.33) and $\Psi(s) = \mathcal{L}\{\psi(t)\}$. We have

$$s^2\Psi(s) + \Psi(s) = (s^2+1)\Psi(s) = -4\frac{2}{s^2+4} \implies \Psi(s) = \frac{1}{s^2+1}\cdot(-4)\frac{2}{s^2+4}.$$

Let

$$H(s) = \frac{1}{s^2+1} \implies h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t.$$

Therefore, the solution y(t) of (6.31) is

$$\begin{aligned} y(t) &= \phi(t) + \psi(t) = 2\sin t + (h * g)(t) \\ &= 2\sin t + \int_0^t h(t-\tau)g(\tau)d\tau \\ &= 2\sin t - 4\int_0^t \sin(t-\tau)\sin(2\tau)d\tau. \end{aligned}$$

Using the trigonometric identity

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right) \quad \text{and} \quad \cos(-\theta) = \cos(\theta),$$

we have

$$\begin{aligned}
\int_0^t \sin(t-\tau)\sin(2\tau)d\tau &= \frac{1}{2}\int_0^t \Big(\cos(t-3\tau)-\cos(t+\tau)\Big)d\tau \\
&= \frac{1}{2}\int_0^t \cos(3\tau-t)d\tau - \frac{1}{2}\int_0^t \cos(\tau+t)d\tau \\
&= \frac{1}{6}\int_0^t \cos(3\tau-t)d(3\tau-t) - \frac{1}{2}\int_0^t \cos(\tau+t)d(\tau+t) \\
&= \frac{1}{6}\left(\sin(3\tau-t)\Big|_{\tau=0}^{\tau=t}\right) - \frac{1}{2}\left(\sin(\tau+t)\Big|_{\tau=0}^{\tau=t}\right) \\
&= \frac{1}{6}\left(\sin(2t)-\sin(-t)\right) - \frac{1}{2}\left(\sin(2t)-\sin t\right) \\
&= -\frac{1}{3}\sin(2t) + \left(\frac{1}{6}+\frac{1}{2}\right)\sin t = -\frac{1}{3}\sin(2t)+\frac{2}{3}\sin t.
\end{aligned}$$

As a result, the solution $y(t)$ of (6.31) is

$$y(t) = 2\sin t - \frac{8}{3}\sin t + \frac{4}{3}\sin(2t) = -\frac{2}{3}\sin t + \frac{4}{3}\sin(2t).$$

□

Example 6.6.4. Express the solution of the given initial-value problem in terms of a convolution integral:

$$y'' + 3y' + 2y = \cos(at), \quad y(0) = 1, \quad y'(0) = 0. \tag{6.34}$$

Solution. First, we split the original problem into two sub-problems:

$$\phi'' + 3\phi' + 2\phi = 0, \quad \phi(0) = 1, \quad \phi'(0) = 0 \tag{6.35}$$

and

$$\psi'' + 3\psi' + 2\psi = \cos(at), \quad \psi(0) = 0, \quad \psi'(0) = 0. \tag{6.36}$$

Here, the problem (6.35) can be solved by the approach introduced in Chapter 3. In particular, we solve the characteristic equation

$$r^2 + 3r + 2 = (r+2)(r+1) = 0 \implies r_1 = -1, \quad r_2 = -2.$$

We obtain

$$\phi(t) = c_1 e^{-t} + c_2 e^{-2t} \implies c_1 + c_2 = 1, \quad -c_1 - 2c_2 = 0.$$

Solving for c_1 and c_2, we have $c_1 = 2$ and $c_2 = -1$ and

$$\phi(t) = 2e^{-t} - e^{-2t}.$$

Next, we solve $\psi(t)$. Denote $\Psi(s) = \mathcal{L}\{\psi(t)\}$. Using the Laplace transform, we have

$$(s^2+3s+2)\Psi(s) = \frac{s}{s^2+a^2} \implies \Psi(s) = \frac{s}{(s^2+3s+2)(s^2+a^2)}.$$

By completing the square, we have

$$s^2+3s+2 = s^2+3s+\frac{9}{4}-\frac{1}{4} = \left(s+\frac{3}{2}\right)^2 - \frac{1}{2^2}.$$

Then, we have x

$$\Psi(s) = \underbrace{\frac{1}{\left(s+\frac{3}{2}\right)^2 - \frac{1}{2^2}}}_{=:F(s)} \cdot \underbrace{\frac{s}{s^2+a^2}}_{G(s)}.$$

Note that

$$G(s) = \mathcal{L}\{\cos(at)\}$$

and

$$F(s) = \frac{1}{\left(s+\frac{3}{2}\right)^2 - \frac{1}{2^2}} = 2\frac{1/2}{\left(s+\frac{3}{2}\right)^2 - \frac{1}{2^2}} = 2\mathcal{L}\left\{e^{-3t/2}\sinh\left(\frac{t}{2}\right)\right\}.$$

Here, $\sinh(t) = (e^t - e^{-t})/2$. Denote $f(t) = 2e^{-3t/2}\sinh(t/2)$ and $g(t) = \cos(at)$. Using the convolution theorem, we have

$$\begin{aligned}\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\} = (f*g)(t) &= \int_0^t f(t-\tau)g(\tau)d\tau \\ &= 2\int_0^t e^{-3(t-\tau)/2}\sinh\left(\frac{t-\tau}{2}\right)\cos(a\tau)d\tau.\end{aligned}$$

Hence, the solution $y(t)$ can be written as

$$y(t) = \phi(t) + \psi(t) = 2e^{-t} - e^{-2t} + 2\int_0^t e^{-3(t-\tau)/2}\sinh\left(\frac{t-\tau}{2}\right)\cos(a\tau)d\tau.$$

We remark that one can use the following (indefinite) integration identity

$$\int e^{\alpha\tau}\cos(\beta\tau)d\tau = \frac{e^{\alpha\tau}}{\alpha^2+\beta^2}(\alpha\cos(\beta\tau) + \beta\sin(\beta\tau)) + C$$

to simplify the convolution integral with $(\alpha,\beta) = (1,a)$ and $(\alpha,\beta) = (2,a)$ to calculate $\psi(t)$. As a result, we have

$$\begin{aligned}\psi(t) &= e^{-3t/2}\int_0^t e^{3\tau/2}\left(e^{(t-\tau)/2} - e^{-(t-\tau)/2}\right)\cos(a\tau)d\tau \\ &= e^{-t}\int_0^t e^{\tau}\cos(a\tau)d\tau - e^{-2t}\int_0^t e^{2\tau}\cos(a\tau)d\tau \\ &= e^{-t}\int_0^t e^{\tau}\cos(a\tau)d\tau - e^{-2t}\int_0^t e^{2\tau}\cos(a\tau)d\tau \\ &= e^{-t}\left(\frac{e^{\tau}}{1+a^2}(\cos(a\tau) + a\sin(a\tau)\Big|_{\tau=0}^{\tau=t}\right) - e^{-2t}\left(\frac{e^{2\tau}}{4+a^2}(2\cos(a\tau) + a\sin(a\tau)\Big|_{\tau=0}^{\tau=t}\right) \\ &= \frac{e^{-t}}{1+a^2}\left(e^t\cos(at) + ae^t\sin(at) - 1\right) - \frac{e^{-2t}}{4+a^2}\left(2e^{2t}\cos(at) + ae^{2t}\sin(at) - 2\right) \\ &= \frac{1}{a^2+1}\left(\cos(at) + a\sin(at) - e^{-t}\right) - \frac{1}{a^2+4}\left(2\cos(at) + a\sin(at) - 2e^{-2t}\right).\end{aligned}$$

□

6.7 Exercises

We denote $u_c(t)$ the unit step function at the point $c \geqslant 0$ and $\delta(t-t_0)$ the Dirac delta function at the point t_0.

1. Find the Laplace transform $\mathcal{L}\{f(t)\}$ of the given functions $f(t)$.
   (a) $f(t) = te^{at}$, where a is a real constant.
   (b) $f(t) = t^2 \sin(at)$, where a is a real constant.
   (c) $f(t) = \begin{cases} 0, & t < 2, \\ (t-2)^2, & t \geqslant 2. \end{cases}$ **Hint:** we can write $f(t) = u_2(t)(t-2)^2$.
2. Find the functions $f(t)$ such that $\mathcal{L}\{f(t)\}(s) = F(s)$.
   (a) $F(s) = \dfrac{3}{s^2+4}$.
   (b) $F(s) = \dfrac{1-2s}{s^2+4s+5}$.
3. Use the Laplace transform to solve the given initial value problems.
   (a) $y'' + 9y = \cos(2t),\ y(0) = 1,\ y'(0) = 0$.
   (b) $y'' + 3y' + 2y = 1 - u_{10}(t),\ y(0) = 0,\ y'(0) = 0$.
   (c) $y'' + y' + \frac{5}{4}y = (1 - u_\pi(t))\sin(t),\ y(0) = 0,\ y'(0) = 0$.
   (d) $y'' + \frac{1}{2}y' + y = \delta(t-1),\ y(0) = 0,\ y'(0) = 0$.
4. Find the Laplace transform of the given function
$$f(t) = \int_0^t \sin(t-\tau)\cos\tau d\tau$$
   using the convolution theorem.
5. Find the inverse Laplace transform of the given function
$$F(s) = \frac{s}{(s+1)(s^2+4)}$$
   using the convolution theorem.

Exercises

There are 3 questions in this assignment. Answer all. Please write down your name and UIN. The deadline is **11:59 pm (CDT), Nov 6 2022**.

We denote $u_c(t)$ the unit step function at the point $c \geqslant 0$ and $\delta(t-t_0)$ the Dirac delta function at the point t_0.

1. Find the Laplace transform $\mathcal{L}\{f(t)\}$ of the given functions $f(t)$.

   (a) $f(t) = te^{at}$, where a is a real constant.

   (b) $f(t) = t^2 \sin(at)$, where a is a real constant.

   (c) $f(t) = \begin{cases} 0, & t < 2, \\ (t-2)^2, & t \geqslant 2. \end{cases}$ **Hint:** we can write $f(t) = u_2(t)(t-2)^2$.

   (d) (Optional for 5 points) $f(t) = \begin{cases} t & 0 \leqslant t \leqslant 1, \\ 2-t & 1 < t \leqslant 2, \\ 0 & t > 2. \end{cases}$ **Hint:** can you write $f(t)$ as the sum of linear functions times multiplying by step functions?

2. Find the functions $f(t)$ such that $\mathcal{L}\{f(t)\}(s) = F(s)$.

   (a) $F(s) = \dfrac{3}{s^2+4}$.

   (b) $F(s) = \dfrac{1-2s}{s^2+4s+5}$.

3. Use the Laplace transform to solve the given initial value problems.

   (a) $y'' + 9y = \cos(2t),\ y(0) = 1,\ y'(0) = 0.$

   (c) $y'' + y' + \frac{5}{4}y = (1 - u_\pi(t))\sin(t),\ y(0) = 0,\ y'(0) = 0.$

Selected Reference Solution

1. (a) $\mathcal{L}\{te^{at}\} = \dfrac{1}{(s-a)^2}$.

   (b) $\mathcal{L}\{t^2\sin(at)\} = -\dfrac{2a(a^2-3s^2)}{(s^2+a^2)^3}$.

   (c) $\mathcal{L}\{u_2(t)(t-2)^2\} = 2e^{-2s}s^{-3}$.

2. (a) $f(t) = \dfrac{3}{2}\sin(2t)$.

   (b) $f(t) = e^{-2t}(5\sin(t) - 2\cos(t))$.

3. (a) $y(t) = \cos(3t) + \dfrac{1}{3}\left[\sin(3t) * \cos(2t)\right] = \dfrac{1}{5}(\cos(2t) + 4\cos(3t))$.

Chapter 7

Systems of First-Order Linear Equations

In this chapter, we study the system of first-order linear differential equations that arise in many physical problems. To study the system of linear differential equations, we have to utilize some of the elementary aspects of linear algebra to unify the presentation.

7.1 Introduction

System of differential equations arises naturally in problem involving several dependent variables, each of which is a function of the same single independent variable. For instance, if we consider the spring-mass system with two objects with different external forces for each object, we end up with a system of equations: if we denote x_1 the displacement for the first object and x_2 for the second, then we have

$$\begin{aligned} m_1 x_1''(t) &= k_2(x_2 - x_1) - k_1 x_1 + F_1(t) = -(k_1 + k_2)x_1 + k_2 x_2 + F_1(t), \\ m_2 x_2''(t) &= -k_3 x_2 - k_2(x_2 - x_1) + F_2(t) = k_2 x_1 - (k_2 + k_3)x_2 + F_2(t). \end{aligned} \tag{7.1}$$

Here, k_1, k_2, and k_3 are spring constants, and m_1, m_2 are mass of the objects respectively. The external forces for F_1 and F_2 are acting on the objects. This is a second-order system of differential equations.

For higher order equation, we can also rewrite it as a system of first-order equations, which is easier to handle using numerical methods. Almost all codes for generating numerical approximations to solutions of differential equations are capable for systems of first-order equations. The following example demonstrates how easy it is to make the transformation from a second-order differential equation to a system of two first-order differential equations.

Example 7.1.1. Rewrite the following spring-mass system

$$u'' + \frac{1}{8}u' + u = 0$$

as a system of first-order equations.

Solution. We write $x_1 = u$ and $x_2 = u'$. Then it follows that $x_1' = x_2$. Further, $u'' = x_2'$. Then, by substituting for u, u', and u'', we get

$$x_2' + \frac{1}{8}x_2 + x_1 = 0 \iff x_2' = -x_1 - \frac{1}{8}x_2.$$

Thus, x_1 and x_2 satisfy the following system of two first-order equations

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -x_1 - \frac{1}{8}x_2. \end{aligned} \tag{7.2}$$

The governing equation of motion of a spring-mass system

$$mu'' + \gamma u' + ku = F(t)$$

can also be transformed into a system of first-order differential equations in a same manner. □

To transform an arbitrary n-th order equation

$$y^{(n)} = F(t, y, y', y'', \cdots, y^{(n-1)})$$

into a system of n first-order differential equations, we extend the method in the above example by introducing $x_1, x_2, \cdots, x_n$, such that

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad \cdots, \quad x_n = y^{(n-1)}.$$

It then follows that

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ &\vdots \\ x_{n-1}' &= x_n, \\ x_n' &= F(t, x_1, x_2, \cdots, x_n). \end{aligned} \tag{7.3}$$

More generally, we consider the system of first-order equations as follows:

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \cdots, x_n), \\ x_2' &= F_2(t, x_1, x_2, \cdots, x_n), \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \cdots, x_n). \end{aligned} \tag{7.4}$$

The above formulation includes almost all cases of interest. Much of the more advanced theory of differential equations is devoted to such systems. A *solution* of the system (7.4) (defined on some interval) consists of n functions

$$x_1 = \phi_1(t), \quad x_2 = \phi_2(t), \quad \cdots, \quad x_n = \phi(t)$$

where each function is differentiable at all points in the interval and satisfy the system of equations (7.4). In addition to the given system of equations, there may also be given n initial conditions of the form

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \cdots, \quad x_n(t_0) = x_n^0,$$

where t_0 is a specified value of t in the interval, and x_i^0's are prescribed numbers. This system of equations and the initial conditions together form an *initial-value problem*.

7.2 Matrices

For both theoretical and computational reasons, it is advisable to bring some of the results of linear algebra to bear on the initial value problem for a system of linear differential equations. In this section, we present a brief summary of the facts that will be needed later. More details can be found in any elementary book on linear algebra.

Definition 7.2.1 (Matrix)**.** An $n \times m$ matrix A is a rectangular array of elements with n rows and m columns in which not only is the value of an element important, but also its position in the array. That is,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,m-1} & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2,m-1} & a_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,m-1} & a_{nm} \end{pmatrix}.$$

We sometimes denote $A = (a_{ij})$. If $n = m$, then the matrix is called a *square matrix of order* n.

Definition 7.2.2 (Transpose)**.** Given an $n \times m$ matrix $A = (a_{ij})$, we define its transpose as an $m \times n$ matrix, denoted A^T, as follows:

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,m-1} & a_{2,m-1} & \cdots & a_{n,m-1} \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}.$$

Definition 7.2.3 (Adjoint)**.** Given an $n \times m$ matrix $A = (a_{ij})$, we define its adjoint as an $m \times n$ matrix, denoted as A^*, as follows:

$$A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{n1}} \\ \overline{a_{12}} & \overline{a_{22}} & \cdots & \overline{a_{n2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1,m-1}} & \overline{a_{2,m-1}} & \cdots & \overline{a_{n,m-1}} \\ \overline{a_{1m}} & \overline{a_{2m}} & \cdots & \overline{a_{nm}} \end{pmatrix}.$$

Here, $\overline{a}$ is the complex conjugate of any complex number a.

Example 7.2.4. Let A be a matrix as follows:

$$A = \begin{pmatrix} 3 & 2-i \\ 4+3i & -5+2i \end{pmatrix}.$$

The transpose and adjoint of A are

$$A^T = \begin{pmatrix} 3 & 4+3i \\ 2-i & 5+2i \end{pmatrix} \quad \text{and} \quad A^* = \begin{pmatrix} 3 & 4-3i \\ 2+i & 5-2i \end{pmatrix}.$$

Definition 7.2.5 (Equivalence of matrices)**.** Two matrices A and B are equal if they have the same number of rows and columns, say $n \times m$, and if $a_{ij} = b_{ij}$, for each $i = 1, \cdots, n$, and $j = 1, \cdots, m$.

Example 7.2.6. The following matrix

$$A = \begin{pmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{pmatrix}$$

has two rows and three columns, so it is of size 2×3. Its entries are described by $a_{11} = 2$, $a_{12} = -1$, $a_{13} = 7$, $a_{21} = 3$, $a_{22} = 1$, and $a_{23} = 0$. In this case, the transpose of A is

$$A^T = \begin{pmatrix} 2 & 3 \\ -1 & 1 \\ 7 & 0 \end{pmatrix}.$$

Note that A^T is of size 3×2. Based on Definition 7.2.5, we have

$$A = \begin{pmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 2 & 3 \\ -1 & 1 \\ 7 & 0 \end{pmatrix} = A^T$$

because they differ in dimension.

In general, a matrix and its transpose are of same size only if this matrix is a square matrix. We have the following definition if the matrix itself is equal to its transpose.

Definition 7.2.7 (Symmetric)**.** A matrix M is called symmetric if $M^T = M$.

Definition 7.2.8 (Self-adjoint)**.** A matrix M is called self-adjoint if $M^* = M$.

Example 7.2.9. The following matrix

$$M = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}$$

is symmetric.

Two important operations performed on matrices are the sum of two matrices and the multiplication of a matrix by a real number (scalar).

Definition 7.2.10 (Sum of two matrices)**.** If A and B are both $n \times m$ matrices, then the sum of A and B, denoted $A + B$, is the $n \times m$ matrix whose entries are $a_{ij} + b_{ij}$ for each $i = 1, \cdots, n$, and $j = 1, \cdots, m$.

Definition 7.2.11 (Scalar multiplication)**.** If A is an $n \times m$ matrix and λ is a real number, then the scalar multiplication of λ and A, denoted λA, is the $n \times m$ matrix whose entries are λa_{ij} for each $i = 1, \cdots, n$, and $j = 1, \cdots, m$.

Example 7.2.12. Determine $A + B$ and λA when

$$A = \begin{pmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 2 & -8 \\ 0 & 1 & 6 \end{pmatrix}, \quad \text{and} \quad \lambda = -2.$$

Solution. We have

$$A + B = \begin{pmatrix} 2+4 & -1+2 & 7-8 \\ 3+0 & 1+1 & 0+6 \end{pmatrix} = \begin{pmatrix} 6 & 1 & -1 \\ 3 & 2 & 6 \end{pmatrix}$$

and

$$\lambda A = \begin{pmatrix} -2(2) & -2(-1) & -2(7) \\ -2(3) & -2(1) & -2(0) \end{pmatrix} = \begin{pmatrix} -4 & 2 & -14 \\ -6 & -2 & 0 \end{pmatrix}.$$

□

We have the following general properties for matrix addition and scalar multiplication. These properties are sufficient to classify the set of all $n \times m$ matrices with real entries as a **vector space** over the field of real numbers. We let O denote a matrix all of whose entries are 0 and $-A$ denote the matrix whose entries are $-a_{ij}$.

Theorem 7.2.13. Let A, B, C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

$$\begin{array}{ll} \text{(i) } A+B=B+A; & \text{(ii) } (A+B)+C=A+(B+C); \\ \text{(iii) } A+O=O+A=A; & \text{(iv) } A+(-A)=-A+A=O; \\ \text{(v) } \lambda(A+B)=\lambda A+\lambda B; & \text{(vi) } (\lambda+\mu)A=\lambda A+\mu A; \\ \text{(vii) } \lambda(\mu A)=\lambda\mu A; & \text{(viii) } 1A=A. \end{array}$$

All these properties follow from similar results concerning the real numbers.

We introduce the notion of *row* and *column* vectors.

Definition 7.2.14 (Vector)**.** The $1 \times n$ matrix

$$x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

is called an n-dimensional row vector, and the $n \times 1$ matrix

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

is called an n-dimensional column vector. We sometimes denote column vector as $y = (y_1, y_2, \cdots, y_n)^T$. A n-dimensional (column or row) vector $\mathbf{0}_n$ with all entries being zero is called n-dimensional zero vector. We denote $\mathbb{R}^n$ the collection of all n-dimensional column vectors.

Example 7.2.15. Any $n \times n$ matrix A can be understood as an $n \times n$ array, or a collection of n-dimensional column vectors:

$$A = \left(\begin{array}{c|c|c|c} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array}\right).$$

Each column of the matrix A is a n-dimensional vector.

Definition 7.2.16. The set of n-dimensional column vectors $\{e_i\}_{i=1}^n$:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad \text{and} \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

is called a natural basis of $\mathbb{R}^n$. We remark that any n-dimensional column vector $x = (x_i)_{i=1}^n$ can be written in the linear combination of the basis vectors $\{e_i\}_{i=1}^n$ as follows:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = x_1 e_1 + x_2 e_2 + \cdots x_n e_n.$$

Definition 7.2.17 (Identity matrix of order n)**.** We define

$$I_n := (e_1\ e_2 \cdots e_n) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

to be the identity matrix of order n.

Definition 7.2.18 (Matrix-vector multiplication)**.** Given an $n \times m$ matrix $A = (a_{ij})$ and an m-dimensional column vector $y = (y_1, y_2, \cdots, y_m)^T$, the n-dimensional column vector is defined to be their product $x = Ay$: for each $i = 1, \cdots, n$,

$$x = Ay = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{where} \quad x_i = \sum_{j=1}^{m} a_{ij} y_j.$$

One can understand the matrix-vector multiplication from another point of view: the new vector $x = Ay$ is a linear combination of the columns of the matrix A. That is,

$$x = Ay = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = y_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + y_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + y_m \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}.$$

Remark. One may easily check that $I_n x = x$ for any n-dimensional vector $x \in \mathbb{R}^n$.

Example 7.2.19. Let A be a 2×3 matrix as in Example 7.2.6 and $y = (1, 2, 3)^T$. Then, their product is

$$x = Ay = \begin{pmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 - 1 \cdot 2 + 7 \cdot 3 \\ 3 + 1 \cdot 2 + 0 \cdot 3 \end{pmatrix} = \begin{pmatrix} 21 \\ 5 \end{pmatrix}.$$

Moreover, we can understand this operation from another point of view:

$$x = Ay = 1 \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 7 \\ 0 \end{pmatrix} = \begin{pmatrix} 21 \\ 5 \end{pmatrix}.$$

We can use this matrix-vector multiplication to define general matrix-matrix multiplication.

Definition 7.2.20 (Matrix-matrix multiplication)**.** Let A be an $n \times m$ matrix and B an $m \times p$ matrix. The **matrix product** of A and B, denoted AB, is an $n \times p$ matrix $C = AB$ whose entries c_{ij} are

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{im} b_{mj},$$

for each $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, p$.

The computation of c_{ij} can be viewed as the multiplication of the entries of the i-th row of A with corresponding entries in the j-th column of B, followed by a summation; that is,

$$c_{ij} = (a_{i1}, a_{i2}, \cdots, a_{im}) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix}.$$

This explains why the number of columns of A must equal the number of rows of B for the product AB to be defined. Moreover, one can also understand matrix-matrix multiplication in the following sense:

$$AB = A \left(\begin{array}{c|c|c|c} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{array}\right) = A\,(b_1\ b_2\ \cdots\ b_p) = (Ab_1\ Ab_2\ \cdots\ Ab_p)\,,$$

where b_j is the j-th column of the matrix B for $j = 1, 2, \cdots, p$. The following example should serve to clarify the matrix multiplication process.

Example 7.2.21. Determine all possible produces of the matrices:

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 1 & 0 & 1 \\ -1 & 3 & 2 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}.$$

Solution. The size of the matrices are

$$A : 3 \times 2, \quad B : 2 \times 3, \quad C : 3 \times 4, \quad \text{and} \quad D : 2 \times 2.$$

The product that can be defined, and their dimensions, are:

$$AB : 3 \times 3, \quad BA : 2 \times 2, \quad AD : 3 \times 2, \quad BC : 2 \times 4, \quad DB : 2 \times 3, \quad \text{and} \quad DD = D^2 : 2 \times 2.$$

These products are

$$AB = \begin{pmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{pmatrix}, \quad BA = \begin{pmatrix} 4 & 1 \\ 10 & 15 \end{pmatrix}, \quad AD = \begin{pmatrix} 7 & -5 \\ 1 & 0 \\ 9 & -5 \end{pmatrix},$$

$$BC = \begin{pmatrix} 2 & 4 & 0 & 3 \\ 7 & 8 & 6 & 4 \end{pmatrix}, \quad DB = \begin{pmatrix} -1 & 0 & -3 \\ 1 & 1 & -4 \end{pmatrix}, \quad \text{and} \quad D^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

□

Notice that although the matrix products AB and BA are both defined, their results are very different: they do not even have the same dimension. We say that the matrix product operation is *not commutative*, that is, products in reverse order can differ. This is the case even when both products are defined and are of the same dimension. Almost any example will show this, for example,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{whereas} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Certain important operations involving matrix product do hold, however, as indicated in the following result.

Theorem 7.2.22. Let A be $n \times m$ matrix, B be an $m \times k$ matrix, C be a $k \times p$ matrix, D be an $m \times k$ matrix, and λ be a real number. The following properties hold:

- $A(BC) = (AB)C$;
- $A(B + D) = AB + AD$;
- $\lambda(AB) = A(\lambda B)$.

Recall that matrices that have the same number of rows as columns are called square matrix and they are important in applications. We summarize the terminology used in this Chapter.

Definition 7.2.23. Let A be a matrix. Then,

- A is called a **square** matrix if it has the same number of rows as columns.
- A is called a **diagonal** matrix if it is a square matrix with $a_{ij} = 0$ if $i \neq j$.

By definition, the **identity matrix of order** n is a diagonal matrix whose diagonal entries are all 1s. When the size of I_n is clear, this matrix is generally written simply as I. (See Definition 7.2.17)

Example 7.2.24. Consider the identity matrix of order three,

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If A is any 3×3 matrix, then

$$AI_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A.$$

One can also check that $I_3A = A$. The identity matrix I_n commutes with any $n \times n$ matrix A; that is, the order of multiplication does not matter,

$$I_nA = AI_n = A.$$

Keep in mind that this property of commuting is not true in general (i.e., $AB \neq BA$ is not true in general), even for square matrices.

Definition 7.2.25 (Upper and lower triangular)**.** An **upper-triangular** $n \times n$ matrix $U = (u_{ij})$ has, for each $j = 1, 2, \cdots, n$, the entries

$$u_{ij} = 0, \quad \text{for each } i = j + 1, j + 2, \cdots, n;$$

and a **lower-triangular** $n \times n$ matrix $L = (\ell_{ij})$ has, for each $j = 1, 2, \cdots, n$, the entries

$$\ell_{ij} = 0, \quad \text{for each } i = 1, 2, \cdots, j - 1.$$

We remark that a diagonal matrix is both upper triangular and lower triangular because its only nonzero entries must lie on the main diagonal.

Definition 7.2.26 (Invertible matrix)**.** Assume that a square matrix A of order n is given. If there exist a set of n-dimensional vectors $\{v_i\}_{i=1}^n$ such that

$$Av_i = e_i \quad \text{for all } i = 1, 2, \cdots, n,$$

where e_i's are the natural basis of $\mathbb{R}^n$. Then, the matrix A is called invertible (or nonsingular) and the matrix containing all vectors $\{v_i\}_{i=1}^n$ is called the inverse of A, denoting A^{-1}. That is,

$$A^{-1} = (v_1 \ v_2 \ \cdots \ v_n).$$

Moreover, we have $AA^{-1} = A^{-1}A = I_n$. A matrix without an inverse is called **singular** (*or noninvertible*).

The following properties regarding matrix inverses follow from the definition above.

Theorem 7.2.27. For any invertible $n \times n$ matrix A:

- A^{-1} is unique;
- A^{-1} is nonsingular and $(A^{-1})^{-1} = A$;
- If B is also a nonsingular $n \times n$ matrix, then $(AB)^{-1} = B^{-1}A^{-1}$.

Example 7.2.28 (Magic Formula for 2×2)**.** Let A be 2×2 as follows:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let B be another 2×2 matrix such that

$$B = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Verify that $B = A^{-1}$ (i.e. $AB = BA = I_2$).

Solution. We verify one way $AB = I_2$ (the another way can also be verified similarly). By straightforward computation, we have

$$AB = \frac{1}{ad - bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} ad - bc & -ab + ba \\ cd - dc & -bc + ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

□

Example 7.2.29. Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{9} \begin{pmatrix} -2 & 5 & -1 \\ 4 & -1 & 2 \\ -3 & 3 & 3 \end{pmatrix}.$$

One can check that $B = A^{-1}$.

Recall the transpose of a given matrix A, denoted A^T. The proof of the next result follows directly from the definition of the transpose.

Theorem 7.2.30. The following operations involving the transpose of a matrix hold whenever the operation is possible:

- $(A^T)^T = A$;
- $(AB)^T = B^T A^T$;

- $(A+B)^T = A^T + B^T$;
- If A^{-1} exists, then $(A^{-1})^T = (A^T)^{-1}$.

We sometimes need to consider vectors of matrices whose elements are functions of t. We write

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \text{and} \quad A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & \cdots & a_{2n}(t) \\ \vdots & & \vdots \\ a_{m1}(t) & \cdots & a_{mn}(t) \end{pmatrix},$$

respectively. The matrix $A(t)$ is said to be continuous at $t = t_0$ or on an interval $I = [\alpha, \beta]$ if each element of A is a continuous function at the given point or on the given interval I. Similarly, $A(t)$ is said to be differentiable if each of its elements is differentiable and we denote

$$A'(t) = \frac{dA}{dt} = \left(\frac{da_{ij}}{dt}\right).$$

Note that $A'(t)$ is a matrix of the same size as $A(t)$, where we take the derivative elementwise. In the same way, the integral of a matrix function is defined as

$$\int_a^b A(t)\, dt = \left(\int_a^b a_{ij}(t)\, dt\right).$$

It is again a matrix of the same size as $A(t)$, where we take integration elementwise.

Example 7.2.31. Let $A(t)$ be given as follows:

$$A(t) = \begin{pmatrix} \sin t & t \\ 1 & \cos t \end{pmatrix}.$$

Then, we have

$$A'(t) = \begin{pmatrix} \cos t & 1 \\ 0 & -\sin t \end{pmatrix} \quad \text{and} \quad \int_0^\pi A(t) dt = \begin{pmatrix} 2 & \pi^2/2 \\ \pi & 0 \end{pmatrix}.$$

Many of the rules of elementary calculus extend easily to matrix functions. Let $A = A(t)$ and $B = B(t)$ be matrix functions and C be a constant matrix. Then, we have

1. $(CA)' = CA'(t)$;
2. $(A+B)' = A'(t) + B'(t)$; and
3. $(AB)' = A(t)B'(t) + A'(t)B(t)$.

In the first and the third equations, care must be taken in each term to avoid interchanging the order of multiplication.

7.3 Systems of Linear Algebraic Equations

In this section, we review some results from linear algebra that will be used in this chapter.

<u>System of linear algebraic equations</u> A set of n linear algebraic equation in n variables

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1,\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2,\\
&\vdots\\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

can be written as

$$A\mathbf{x} = \mathbf{b}$$

where the $n \times n$ matrix A and the n-dimensional vector $\mathbf{b}$ are given. The vector $\mathbf{x}$ is to be determined. If $\mathbf{b} = \mathbf{0}$, the system is said to be *homogeneous*; otherwise, it is *nonhomogeneous.*

If the matrix A is invertible, then the solution vector $\mathbf{x}$ is $\mathbf{x} = A^{-1}\mathbf{b}$. In particular, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$ if A is invertible.

When A is singular, the homogeneous system has infinitely many nonzero solutions in addition to the trivial solution. The situation for the nonhomogeneous system is more complicated. This system has no solution unless $\mathbf{b}$ satisfies a certain further condition.

Example 7.3.1. Solve the system of equations

$$\begin{aligned}
x_1 - 2x_2 + 3x_3 &= 7,\\
-x_1 + x_2 - 2x_3 &= -5,\\
2x_1 - x_2 - x_3 &= 4.
\end{aligned}$$

Solution. To solve the equation effectively, we form the so-called *augmented matrix*

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 7\\ -1 & 1 & -2 & -5\\ 2 & -1 & -1 & 4 \end{array}\right)$$

by adjoining the right-hand side vector to the coefficient matrix as an additional column. In general, we can use the *Gaussian elimination* method to solve the system of linear equations. That is, we perform several row operators to turn the left-hand side matrix into a form that is easier to solve. □

<u>Linear dependence and independence.</u> A collection of k vectors $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(k)}$ is said to be **linearly dependent** if there exists a set of real or complex numbers $c_1, \cdots, c_k$ at least one of which is nonzero, such that

$$c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0}.$$

On the other hand, if the only values of the coefficients $c_1, \cdots, c_k$ are $c_1 = \cdots = c_k = 0$, then $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(k)}$ are said to be **linearly independent**.

Example 7.3.2. Determine whether the vectors

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1\\ 2\\ -1 \end{pmatrix}, \quad \mathbf{x}^{(1)} = \begin{pmatrix} 2\\ 1\\ 3 \end{pmatrix}, \quad \mathbf{x}^{(3)} = \begin{pmatrix} -4\\ 1\\ -11 \end{pmatrix}$$

are linearly independent or dependent.

Solution. To determine whether $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$ are linearly dependent or not, we seek constants c_1, c_2, and c_3 such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}.$$

□

<u>Eigenvalues and eigenvectors.</u> The equation

$$A\mathbf{x} = \mathbf{y}$$

can be viewed as a linear transformation that maps a given vector $\mathbf{x}$ into a new vector $\mathbf{y}$. Vectors that are transformed into multiples of themselves are important in many applications, including finding solutions to system of first-order linear differential equations with constant coefficients.

To find such a vector, we set $\mathbf{y} = \lambda\mathbf{x}$, where λ is a scalar and we seek solutions of the equation

$$A\mathbf{x} = \lambda\mathbf{x} \iff (A - \lambda I_n)\mathbf{x} = \mathbf{0}.$$

The latter equation has nonzero solutions if and only if λ is chosen so that

$$\det(A - \lambda I_n) = 0. \tag{7.5}$$

Here, I_n is the identity matrix of order n. This equation is a polynomial equation of degree n in λ and is called the **characteristic equation** of the matrix A. Such values of λ satisfying (7.5) is called the eigenvalues of the matrix A and the nonzero vectors $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ are called eigenvectors corresponding to that eigenvalue λ.

Example 7.3.3. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}.$$

Solution. The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation $(A - \lambda I_2)\mathbf{x} = \mathbf{0}$ or

$$\begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The eigenvalues are the roots of the following **characteristic equation**:

$$\det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0.$$

Thus the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$. To find the eigenvectors, we replace λ by each of the eigenvalues in turn and find the values of x_1 and x_2. For $\lambda = 2$, we have

$$\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, each row of this vector equation leads to the condition $x_1 - x_2 = 0$, so x_1 and x_2 are equal but their values are not determined. If $x_1 = c$, then $x_2 = c$ also, and the eigenvector $\mathbf{x}^{(1)}$ is

$$\mathbf{x}^{(1)} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c \neq 0.$$

Thus, for the eigenvalue $\lambda_1 = 2$, there is an infinite family of eigenvectors, indexed by the arbitrary constant c. We choose a single number of this family as a representative of the rest; in this example, it seems simplest to let $c = 1$. Then, we write

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and remember that any nonzero multiple of this vector is also an eigenvector.

Now, setting $\lambda = -1$ we have

$$\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Again, we obtain a single condition on x_1 and x_2, namely, $4x_1 - x_2 = 0$. Thus, the eigenvector corresponding to the eigenvalue $\lambda_2 = -1$ is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

or any nonzero multiple of this vector. □

Example 7.3.4. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Solution. The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation $(A - \lambda I_3)\mathbf{x} = \mathbf{0}$ or

$$\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The eigenvalues are the roots of the equation

$$\det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 2 = (\lambda - 2)(\lambda + 1)^2 = 0.$$

The roots are $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = -1$. To find the eigenvector $\mathbf{x}^{(1)}$ corresponding to the eigenvalue λ_1, we substitute $\lambda = 2$ and solve

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We can use elementary row operations to reduce this to the equivalent system

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving this system yields the eigenvector

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda = -1$, it reduces to the single equation

$$x_1 + x_2 + x_3 = 0.$$

Thus values for two of the quantities x_1, x_2, and x_3 can be chosen arbitrarily, and the third is determined. For example, if $x_1 = c_1$ and $x_2 = c_2$, then $x_3 = -c_1 - c_2$. In vector notation, we have

$$\mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \\ -c_1 - c_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

For example, by choosing $c_1 = 1$ and $c_2 = 0$, we obtain the eigenvector

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Any nonzero multiple of $\mathbf{x}^{(2)}$ is also an eigenvector, but a second linearly independent eigenvector can be found by making another choice of c_1 and c_2, for instance $c_1 = 0$ and $c_2 = 1$. In this case, we obtain

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

which is linearly independent of $\mathbf{x}^{(2)}$. Therefore, in this example, two linearly independent eigenvectors are associated with the eigenvalue $\lambda = -1$. □

7.4 Basic Theory of Systems of First-Order Linear Equations

In this section, we introduce some theoretical results of system of first-order linear equations.

We consider the following a system of n first-order linear equations

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}x_n + g_1(t), \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}x_n + g_2(t), \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}x_n + g_n(t). \end{aligned} \tag{7.6}$$

or equivalently

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t). \tag{7.7}$$

A vector $\mathbf{x} = \mathbf{x}(t)$ is said to be a solution of (7.7) if its components satisfy the system of equations (7.6). When $\mathbf{g}(t) = \mathbf{0}$, i.e.,

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \tag{7.8}$$

the system is called **homogeneous**. Just as before, once the homogeneous equation has been solved, there are several methods that can be used to solve the nonhomogeneous one. We summarize some useful theoretical results of homogeneous system of first-order linear equations.

- If $\mathbf{x}$ and $\mathbf{y}$ are solutions of (7.8), so is their linear combination $c_1\mathbf{x} + c_2\mathbf{y}$ for any constants c_1 and c_2.

- If $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}$ are solutions of (7.8), then the determinant of the following matrix
$$\mathbf{X}(t) = \left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}\right]$$
is called the Wronskian of the n solutions and is denoted as
$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}] = \det(\mathbf{X}(t)).$$

- If $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}$ are linearly independent solutions of (7.8), then each solution $\mathbf{x}(t)$ of (7.8) can be expressed as
$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)}$$
in exactly one way. We call this is a **general solution** of (7.8). The set $\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}\}$ is called a **fundamental set of solutions**.

- Let $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}$ be the solutions of the system (7.8) that satisfy the initial conditions
$$\mathbf{x}^{(1)}(t_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{x}^{(2)}(t_0) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \quad \mathbf{x}^{(n)}(t_0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$
for some given t_0, then $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}$ form a fundamental set of solutions of (7.8).

- Let $\mathbf{x}(t)$ be the solution of (7.8); If $\mathbf{x}(t) = \mathbf{u}(t) + \mathbf{v}(t)$ is a complex-valued solution, then its real part $\mathbf{u}(t)$ and its imaginary part $\mathbf{v}(t)$ are also solutions of (7.8).

7.5 Homogeneous Linear Systems with Constant Coefficients

We start to find solutions of the system of equation with constant coefficients
$$\mathbf{x}' = A\mathbf{x}, \tag{7.9}$$
where A is a constant $n \times n$ matrix. In this chapter, we mainly focus on $n = 2$ and $n = 3$. When $n = 1$, the system reduces to a single first-order equation
$$x' = ax$$
where solution is $x(t) = ce^{at}$ with constant c. For $n >$, we first look at the below example.

Example 7.5.1. Find the general solution of the system
$$\mathbf{x}' = A\mathbf{x}, \quad \text{where } A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}.$$

Solution. The matrix A is **diagonal** and we can write
$$x_1' = 2x_1, \quad x_2' = -3x_2.$$
Each of these equations involves only one of the unknown variables, so we can solve the two equations separately. In this way, we find
$$x_1 = c_1e^{2t}, \quad x_2 = c_2e^{-3t},$$

where c_1 and c_2 are arbitrary constants. Then, by writing the solution in vector form, we have

$$\mathbf{x}(t) = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}.$$

Next, we define

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ e^{-3t} \end{pmatrix},$$

then the Wronskian of these solutions is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = \begin{vmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{vmatrix} = e^{-t} \neq 0.$$

Therefore, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions. □

Let us extend the idea to the general system (7.9) by seeking solutions of the form

$$\mathbf{x} = \mathbf{v}e^{rt}$$

where the constant r and the vector $\mathbf{v}$ are to be determined. Substituting this into the system (7.9), we have

$$r\mathbf{v}e^{rt} = A\mathbf{v}e^{rt} \implies A\mathbf{v} = r\mathbf{v}.$$

That is, r is an eigenvalue of A and $\mathbf{v}$ is the eigenvector of A associated with r.

Example 7.5.2. Find the general solution of the system

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where } A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}.$$

Solution. To find the solutions explicitly, we assume the solution has the form $\mathbf{x} = \mathbf{v}e^{rt}$. We have to find eigenvalues of A and the associated eigenvector. To this aim, we solve

$$(A - rI_2)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We have

$$\begin{vmatrix} 1-r & 1 \\ 4 & 1-4 \end{vmatrix} = (1-r)^2 - 4 = r^2 - 2r - 3 = (r-3)(r+1) = 0.$$

Thus, we have the eigenvalues of A to be $r_1 = 3$ and $r_2 = -1$. When $r = 3$, the two equations reduce to the single equation

$$-2v_1 + v_2 = 0 \iff v_2 = 2v_1.$$

The eigenvector corresponding to $r_1 = 3$ can be taken as

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Similarly, corresponding to $r_2 = -1$, we find that $v_2 = -2v_1$, so the eigenvector can be taken as

$$\mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are

$$\mathbf{x}^{(1)}(t) = \mathbf{v}^{(1)}e^{3t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad \mathbf{x}^{(2)}(t) = \mathbf{v}^{(2)}e^{-t} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

The Wronskian of these solutions is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

Hence, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions, and the general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t},$$

where c_1 and c_2 are arbitrary constants. □

7.6 Complex-Valued Eigenvalues

In this section we consider again the system of first-order linear equations

$$\mathbf{x}' = A\mathbf{x}, \tag{7.10}$$

where the coefficient matrix A is real-valued. If we seek the solutions of the form $\mathbf{x} = \mathbf{v}e^{rt}$, then r and $\mathbf{v}$ are the eigenvalue of A and the corresponding eigenvector. It can be the case that r is of complex value with a real-valued matrix A. If $r_1 = \lambda + i\mu$ is an eigenvalue of A, then $r_2 = \lambda - i\mu$ is also an eigenvalue of A.

Example 7.6.1. Find the general solution of the system

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where } A = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}.$$

Solution. To find a fundamental set of solutions, we assume the solution has the form $\mathbf{x} = \mathbf{v}e^{rt}$ and obtain the set of linear equations

$$(A - rI_2)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We have

$$\begin{vmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{vmatrix} = r^2 + r + \frac{5}{4} = 0.$$

Thus, we have the eigenvalues of A to be

$$r_1 = -\frac{1}{2} + i, \quad r_2 = -\frac{1}{2} - i.$$

A straightforward calculation shows that the corresponding eigenvectors are

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Observe that $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are also complex conjugates. Hence, we obtain a fundamental set of solutions of the system:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1/2+i)t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1/2-i)t}.$$

To obtain a set of real-valued solutions, we can choose the real and imaginary parts of either $\mathbf{x}^{(1)}$ or $\mathbf{x}^{(2)}$. In fact, using the Euler's formula $e^{it} = \cos t + i \sin t$, we have

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-t/2}(\cos t + i \sin t) = \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + i \begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}.$$

Hence, a pair of real-valued solutions is

$$\mathbf{u}(t) = \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix}, \quad \mathbf{v}(t) = \begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}.$$

The Wronskian of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ is

$$W[\mathbf{u}(t), \mathbf{v}(t)] = \begin{vmatrix} e^{-t/2}\cos t & e^{-t/2}\sin t \\ -e^{-t/2}\sin t & e^{-t/2}\cos t \end{vmatrix} = e^{-t}(\cos^2 t + \sin^2 t) = e^{-t} \neq 0.$$

It follows that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ form a fundamental set of (real-valued) solutions. Therefore, the general solution is

$$\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) = c_1 \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}.$$

□

To summarize, for 2×2 system A with real coefficients, let $r_1 = \lambda + i\mu$ be an eigenvalue of A and $\mathbf{v}^{(1)} = \mathbf{a} + i\mathbf{b}$ be the corresponding eigenvector. Then, we have

$$\mathbf{x}^{(1)}(t) = (\mathbf{a} + i\mathbf{b})e^{\lambda t + i\mu t} = (\mathbf{a} + i\mathbf{b})e^{\lambda t}(\cos(\mu t) + i \sin(\mu t)).$$

If we write

$$\begin{aligned} \mathbf{x}^{(1)}(t) &= \underbrace{e^{\lambda t}(\mathbf{a}\cos(\mu t) - \mathbf{b}\sin(\mu t))}_{=:\mathbf{u}(t)} + i\, \underbrace{e^{\lambda t}(\mathbf{a}\sin(\mu t) + \mathbf{b}\cos(\mu t))}_{=:\mathbf{v}(t)}, \\ \mathbf{u}(t) &= e^{\lambda t}(\mathbf{a}\cos(\mu t) - \mathbf{b}\sin(\mu t)), \\ \mathbf{v}(t) &= e^{\lambda t}(\mathbf{a}\sin(\mu t) + \mathbf{b}\cos(\mu t)), \end{aligned} \tag{7.11}$$

then $\mathbf{u}(t)$ and $\mathbf{v}(t)$ form a fundamental set of (real-valued) solutions of the system $\mathbf{x}' = A\mathbf{x}$. In this case, the general solution $\mathbf{x}(t)$ can be written as

$$\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t),$$

where c_1 and c_2 are arbitrary constants.

Example 7.6.2. Find the solution of the system

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where } A = \begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix}$$

given the initial conditions

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution. We first find the eigenvalues and eigenvectors of A. To this aim, we expand the characteristic polynomial $\det(A - \lambda I_2)$ as follows:

$$\det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 5 = \lambda^2 - 4\lambda + 8.$$

Completing the square and setting the characteristic polynomial to be 0, we obtain that

$$\lambda^2 - 4\lambda + 8 = (\lambda - 2)^2 + 4 = 0 \iff \lambda = 2 \pm 2i.$$

Hence, we obtain that A has two complex eigenvalues

$$\lambda_1 = 2 + 2i \quad \text{and} \quad \lambda_2 = 2 - 2i.$$

Next, we find the eigenvectors $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ associated with different eigenvalues λ_1 and λ_2, respectively. For $\lambda_1 = 2 + 2i$, we look for $\mathbf{v}^{(1)}$ such that

$$(A - \lambda_1 I_2)\mathbf{v}^{(1)} = \begin{pmatrix} -1-2i & -5 \\ 1 & 1-2i \end{pmatrix} \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, we can take $v_1^{(1)} = -1 + 2i$ and $v_2^{(1)} = 1$. Thus,

$$\mathbf{v}^{(1)} = \begin{pmatrix} -1+2i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \mathbf{a} + i\mathbf{b}.$$

Then, letting $\lambda = 2$ and $\mu = 2$ in (7.11) with such $\mathbf{a}$ and $\mathbf{b}$ above, we can write $\mathbf{u}(t)$ and $\mathbf{v}(t)$ as follows:

$$\mathbf{u}(t) = e^{2t} \begin{pmatrix} -\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix},$$
$$\mathbf{v}(t) = e^{2t} \begin{pmatrix} \sin(2t) + 2\cos(2t) \\ -\sin(2t) \end{pmatrix}.$$

Hence, with arbitrary constants c_1 and c_2, the general solution $\mathbf{x}(t)$ is

$$\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t).$$

To determine c_1 and c_2, we make use of the initial conditions: for $t = 0$, we have

$$c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \iff \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

As a result, we have

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{x}(t) = \mathbf{u}(t) + \mathbf{v}(t) = e^{2t} \begin{pmatrix} \cos(2t) - \sin(2t) \\ \cos(2t) - \sin(2t) \end{pmatrix}.$$

□

7.7 Fundamental Matrices

Suppose that $\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions for the system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \tag{7.12}$$

then the matrix

$$\mathbf{\Psi}(t) = [\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)]$$

whose columns are the vectors $\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)$ is said to be a **fundamental matrix** for the system (7.12). Note that a fundamental matrix is non-singular since its columns are linearly independent vectors. Each column of the fundamental matrix $\mathbf{\Psi}$ is a solution of (7.12). It follows that $\mathbf{\Psi}$ satisfies the matrix differential equation

$$\mathbf{\Psi}' = \mathbf{P}(t)\mathbf{\Psi}.$$

Example 7.7.1. The system

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where } A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

has two solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t},$$

whose Wronskian is nonzero. Thus, a fundamental matrix for such system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}.$$

The solution of an initial-value problem can be written very compactly in terms of a fundamental matrix. The general solution of equation (7.12) is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)} = \mathbf{\Psi}(t)\mathbf{c},$$

where $\mathbf{c}$ is a constant vector with arbitrary components $c_1, c_2, \cdots, c_n$. For an initial-value problem consisting of the system (7.12) and the initial condition

$$\mathbf{x}(t_0) = \mathbf{x}^0,$$

where $\mathbf{x}^0$ is a given vector, it is only necessary to choose the vector $\mathbf{c}$ so as to satisfy the initial condition. Hence, we have

$$\mathbf{\Psi}(t_0)\mathbf{c} = \mathbf{x}^0 \implies \mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0.$$

Therefore, the solution of the initial-value problem reads:

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0$$

given any fundamental matrix $\mathbf{\Psi}(t)$. Sometimes it is convenient to find a special fundamental matrix denoted by $\mathbf{\Phi}(t)$ such that

$$\mathbf{\Phi}(t_0) = I_n.$$

Then, the general solution of (7.12) is written as

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}^0.$$

We will always reserve the symbol $\mathbf{\Phi}$ to denote the fundamental matrix satisfying the condition $\mathbf{\Phi}(t_0) = I_n$.

Example 7.7.2. For the system

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where } A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix},$$

find the fundamental matrix $\mathbf{\Phi}$ such that $\mathbf{\Phi}(0) = I_2$.

Solution. Recall that the general solution of the system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

We have to find two solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ such that

$$\mathbf{x}^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $\mathbf{x}^{(1)}$ be

$$\mathbf{x}^{(1)}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

Then,

$$\mathbf{x}^{(1)}(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies c_1 = c_2 = \frac{1}{2}.$$

Hence, we have

$$\mathbf{x}^{(1)}(t) = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}\right] = \begin{pmatrix} \dfrac{e^{3t}}{2} + \dfrac{e^{-t}}{2} \\ e^{3t} - e^{-t} \end{pmatrix}.$$

Similarly, let $\mathbf{x}^{(2)}$ be

$$\mathbf{x}^{(2)}(t) = d_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + d_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

Then,

$$\mathbf{x}^{(1)}(0) = d_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + d_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies d_1 = \frac{1}{4}, \quad d_2 = -\frac{1}{4}.$$

Hence, we have

$$\mathbf{x}^{(1)}(t) = \frac{1}{4}\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} - \frac{1}{4}\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} = \begin{pmatrix} \dfrac{e^{3t}}{4} - \dfrac{e^{-t}}{4} \\ \dfrac{e^{3t}}{2} + \dfrac{e^{-t}}{2} \end{pmatrix}.$$

Therefore, we have

$$\mathbf{\Phi}(t) = \begin{pmatrix} \dfrac{e^{3t}}{2} + \dfrac{e^{-t}}{2} & \dfrac{e^{3t}}{4} - \dfrac{e^{-t}}{4} \\ e^{3t} - e^{-t} & \dfrac{e^{3t}}{2} + \dfrac{e^{-t}}{2} \end{pmatrix}.$$

□

7.8 Repeated Eigenvalues

We conclude our consideration of the linear homogeneous system of differential equations with constant coefficients

$$\mathbf{x}' = A\mathbf{x} \tag{7.13}$$

with a discussion of the case in which the matrix A has a repeated eigenvalue.

Example 7.8.1. Find the general solution of the system

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where } A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

Solution. The eigenvalues r and eigenvectors $\mathbf{v}$ satisfy the equation

$$(A - rI_2)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The eigenvalues are the roots of the equation

$$\det(A - rI_2) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = r^2 - 4r + 4 = (r-2)^2 = 0.$$

Thus, two eigenvalues are $r_1 = r_2 = 2$; that is, the eigenvalue 2 is a repeated one. To determine the eigenvectors, we return to the equation with $r = 2$ and this gives

$$v_1 + v_2 = 0.$$

Hence, the eigenvector corresponding to $r = 2$ is

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

or any nonzero multiple of this vector. Observe that there is only one linearly independent eigenvector associated with the double eigenvalue.

To solve the system, we find that one of the solutions of the system is

$$\mathbf{x}^{(1)} = \mathbf{v}^{(1)} e^{r_1 t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

To find the second (linearly-independent) solution, we assume that the other solution has the form

$$\mathbf{x} = \mathbf{v} t e^{2t} + \mathbf{w} e^{2t},$$

where $\mathbf{v}$ and $\mathbf{w}$ are to be determined. Upon substituting this expression for $\mathbf{x}$ in the equation, we obtain

$$2\mathbf{v} t e^{2t} + (\mathbf{v} + 2\mathbf{w}) e^{2t} = A(\mathbf{v} t e^{2t} + \mathbf{w} e^{2t}).$$

Equating coefficients of te^{2t} and e^{2t} on each side of the above equation, we have

$$2\mathbf{v} = A\mathbf{v} \quad \text{and} \quad \mathbf{v} + 2\mathbf{w} = A\mathbf{w}$$

for the determination of $\mathbf{v}$ and $\mathbf{w}$. Then, we can see that $\mathbf{v}$ is an eigenvector of A associated with the eigenvalue $r = 2$. We can choose $\mathbf{v} = \mathbf{v}^{(1)}$. Then, the vector $\mathbf{w}$ satisfies

$$(A - 2I_2)\mathbf{w} = \mathbf{v} \iff \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence, we have $w_1 + w_2 = -1$ so if $w_1 = k$, where k is arbitrary, then $w_2 = -1 - k$. Then, we obtain

$$\mathbf{w} = \begin{pmatrix} k \\ -1-k \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, the solution $\mathbf{x}$ can be written as

$$\mathbf{x} = \mathbf{v}te^{2t} + \mathbf{w}e^{2t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

The last term is merely a multiple of the first solution $\mathbf{x}^{(1)}$ and may be ignored, but the first two terms constitute a new solution:

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} = \begin{pmatrix} te^{2t} \\ -te^{2t} - e^{2t} \end{pmatrix}.$$

One can verify that

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = \begin{vmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{vmatrix} = -e^{4t} \neq 0.$$

Therefore, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions of the system. Then, the general solution is

$$\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \right].$$

□

In general, consider the system

$$\mathbf{x}' = A\mathbf{x},$$

and suppose that $r = \rho$ is a double eigenvalue of A, but that there is only one corresponding eigenvector $\mathbf{v}$ (i.e. $A\mathbf{v} = \rho\mathbf{v}$). Then, one solution is

$$\mathbf{x}^{(1)} = \mathbf{v}e^{\rho t}.$$

The second solution is of the form

$$\mathbf{x}^{(2)} = \mathbf{v}te^{\rho t} + \mathbf{w}e^{\rho t}.$$

The vector $\mathbf{w}$ satisfies

$$(A - \rho I)\mathbf{w} = \mathbf{v}.$$

Example 7.8.2. Find the solution of the system

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where } A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$$

with the initial conditions

$$\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution. The eigenvalues λ's of A satisfies

$$0 = \det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{vmatrix} = (\lambda - 1)(\lambda + 7) + 16 = \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2.$$

Hence, there is only one (repeated) eigenvalue $\lambda = -3$ for A. The corresponding eigenvector $\mathbf{v}$ satisfies

$$(A - \lambda I_2)\mathbf{v} = \begin{pmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, we can take

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

One of the solution in the fundamental set of solutions is

$$\mathbf{x}^{(1)} = \mathbf{v}e^{-3t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

Next, we determine the vector $\mathbf{w}$ such that $(A - \lambda I)\mathbf{w} = \mathbf{v}$:

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

If we set $w_1 = k$, then we have $w_2 = -0.25 + k$. Hence, we have

$$\mathbf{w} = \begin{pmatrix} k \\ -0.25 + k \end{pmatrix} = \begin{pmatrix} 0 \\ -0.25 \end{pmatrix} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, another solution in the fundament set of solutions is

$$\mathbf{x}^{(2)} = \mathbf{v}te^{-3t} + \mathbf{w}e^{-3t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 0 \\ -0.25 \end{pmatrix} e^{-3t} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

Since the last term is just a multiple of $\mathbf{x}^{(1)}$, we can drop this term and simply rewrite

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 0 \\ -0.25 \end{pmatrix} e^{-3t}.$$

Then, the fundamental matrix $\mathbf{\Psi}(t)$ reads

$$\mathbf{\Psi}(t) = e^{-3t} \begin{pmatrix} 1 & t \\ 1 & t - 0.25 \end{pmatrix}.$$

The general solution $\mathbf{x}$ can be written as

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

In order to find the values of c_1 and c_2, we make use of the initial conditions and set

$$\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \mathbf{\Psi}(0)\mathbf{c} = \begin{pmatrix} 1 & 0 \\ 1 & -0.25 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

As a result, we have

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -4 \begin{pmatrix} -0.25 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}.$$

Hence, the solution to the system of differential equations is

$$\mathbf{x}(t) = -3\mathbf{x}^{(1)}(t) - 4\mathbf{x}^{(2)}(t) = \begin{pmatrix} -3e^{-3t} - 4te^{-3t} \\ -3e^{-3t} - 4te^{-3t} + e^{-3t} \end{pmatrix} = e^{-3t} \begin{pmatrix} -3 - 4t \\ -2 - 4t \end{pmatrix}.$$

□

7.9 Nonhomogeneous Linear Systems

In this section, we turn to the nonhomogeneous system of linear first-order equations as follows:

$$\mathbf{x}' = P(t)\mathbf{x} + g(t), \tag{7.14}$$

where $\mathbf{x}(t)$ is the $n \times 1$ solution vectors, $P(t)$ is the $n \times n$ matrix, and $g(t)$ are $n \times 1$ continuous vector. By the same argument as in Section 3.5, the general solution of equation (7.14) can be expressed as

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) + \mathbf{v}(t),$$

where $c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ forms a general solution to the corresponding homogeneous system $\mathbf{x}'(t) = P(t)\mathbf{x}(t)$, and $\mathbf{v}(t)$ is a particular solution of the nonhomogeneous system (7.14). We briefly describe several methods for determining $\mathbf{v}(t)$.

<u>Diagonalization</u> We begin with the case when $P(t) = A$ for some constant diagonal matrix A. That is,

$$\mathbf{x}'(t) = A\mathbf{x}(t) + g(t).$$

It is a system that is readily solved since each component of x_i in the solution vector $\mathbf{x}(t)$ satisfies the first-order equation

$$x_i'(t) = a_i x_i + g_i(t).$$

Here, we have

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}.$$

Now we consider the case when A is not diagonal, but it can be diagonalized in the sense that there is some matrix T such that

$$T^{-1}AT = D \iff AT = TD,$$

where D is a diagonal matrix. Here, T's columns contain the eigenvectors of A. Note that if A is a constant matrix, so is T. Hence, if we write $\mathbf{x} = T\mathbf{y}$ for another unknown vector $\mathbf{y} = \mathbf{y}(t)$, then we have

$$\mathbf{x}'(t) = T\mathbf{y}'(t) = AT\mathbf{y}(t) + g(t) \implies T^{-1}T\mathbf{y}'(t) = (T^{-1}AT)\mathbf{y}(t) + T^{-1}g(t) = D\mathbf{y} + T^{-1}g(t).$$

That is, we found that the unknown vector $\mathbf{y}(t)$ satisfies

$$\mathbf{y}'(t) = D\mathbf{y} + T^{-1}g(t),$$

where D is diagonal. Thus, it is readily solved and once we found $\mathbf{y}(t)$, we can multiply $\mathbf{y}$ by the matrix T to recover the desired solution vector $\mathbf{x}(t)$.

Example 7.9.1. Find the general solution of the system

$$\mathbf{x}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = A\mathbf{x} + g(t).$$

Solution. Note that for the matrix A, the eigenvalues are $r_1 = -3$ and $r_2 = -1$ and that the corresponding eigenvectors are

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus the general solution of the homogeneous system is

$$c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-3t}+c_2\begin{pmatrix}1\\1\end{pmatrix}e^{-t}.$$

We can write T matrix to be

$$T=\begin{pmatrix}1&1\\-1&1\end{pmatrix}\implies T^{-1}=\begin{pmatrix}1/2&-1/2\\1/2&1/2\end{pmatrix}.$$

Moreover, we have

$$D=\begin{pmatrix}-3&0\\0&-1\end{pmatrix}.$$

As a result, we have to solve

$$\mathbf{y}'(t)=D\mathbf{y}(t)+T^{-1}g(t)=\begin{pmatrix}-3&0\\0&-1\end{pmatrix}\mathbf{y}(t)+\begin{pmatrix}e^{-t}-.5t\\e^{-t}+1.5t\end{pmatrix}.$$

Hence, we have y_1 and y_2 satisfy the first-order equations

$$\begin{aligned}y_1'+3y_1&=e^{-t}+0.5t,\\ y_2'+y_2&=e^{-t}-1.5t.\end{aligned}\tag{7.15}$$

This implies that

$$\begin{aligned}y_1(t)&=e^{-3t}\int\left[e^{2t}+\frac{t}{2}e^{3t}\right]dt+c_1e^{-3t}=\frac{e^{-t}}{2}+\frac{1}{18}(3t-1)+c_1e^{-3t},\\ y_2(t)&=e^{-t}\int\left[1-\frac{3}{2}te^{t}\right]dt+c_2e^{-t}=te^{-t}+\frac{3}{2}(t-1)+c_2e^{-t}.\end{aligned}\tag{7.16}$$

Hence, we have

$$\mathbf{x}=T\mathbf{y}\implies x_1(t)=y_1(t)+y_2(t)\quad\text{and}\quad x_2(t)=-y_1(t)+y_2(t).$$

□

<u>UNDETERMINED COEFFICIENTS</u> A second way to find a particular solution is the method of undetermined coefficients.

Example 7.9.2. Find a particular solution of the system

$$\mathbf{x}'(t)=\begin{pmatrix}-2&1\\1&-2\end{pmatrix}\mathbf{x}+\begin{pmatrix}2e^{-t}\\3t\end{pmatrix}=A\mathbf{x}+g(t)$$

using the method of undetermined coefficients.

Solution. We write $g(t)$ as follows:

$$g(t)=\begin{pmatrix}2\\0\end{pmatrix}e^{-t}+\begin{pmatrix}0\\3\end{pmatrix}t.$$

Observe that $r=-1$ is an eigenvalue of the coefficient matrix A, and thus we must include both $\mathbf{a}te^{-t}$ and $\mathbf{b}e^{-t}$ in the assumed solution. We assume that the solution has the form

$$\mathbf{x}(t)=\mathbf{a}te^{-t}+\mathbf{b}e^{-t}+\mathbf{c}t+\mathbf{d},$$

where **a**, **b**, **c**, and **d** are vectors to be determined. Substituting this expression into the equation we get

$$-\mathbf{a}te^{-t} + (\mathbf{a} - \mathbf{b})e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + \left[A\mathbf{b} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right]e^{-t} + \left[A\mathbf{c} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right]t + A\mathbf{d}.$$

Equating all the coefficient terms, we obtain that

$$A\mathbf{a} = -\mathbf{a}, \quad A\mathbf{b} = \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad A\mathbf{c} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad A\mathbf{d} = \mathbf{c}.$$

We see that **a** is an eigenvector of A corresponding to the eigenvalue $r = -1$. Thus, we can choose **a** to be

$$a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Next, from the second equation, we can solve for **b** as follows:

$$(A + I_2)\mathbf{b} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\mathbf{b} = \begin{pmatrix} 1-2 \\ 1-0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence, we have

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \text{where} \quad b_1 - b_2 = 1.$$

If we set $b_1 = k$, then $b_2 = k - 1$, and we have

$$\mathbf{b} = \begin{pmatrix} k \\ k-1 \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For the simplest choice, we can take $k = 0$ and let $\mathbf{b} = (0 - 1)^T$. Solving the rest of the equations for **c** and **d** will obtain

$$\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{d} = -\frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Thus, we obtain a particular solution to the system

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

□

<u>Variation of Parameter</u> We can also use the method of variation of parameter to solve the system of equation

$$\mathbf{x}'(t) = A\mathbf{x} + g(t).$$

Here, A can be also depending on t. Assume that a fundamental matrix $\mathbf{\Psi}(t)$ is obtained for the corresponding homogeneous system

$$\mathbf{x}'(t) = A\mathbf{x}(t).$$

Then, we assume the solution to the nonhomogeneous system is

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{u}(t),$$

where $\mathbf{u}(t)$ is a vector to be found. Substituting this expression of **x** to the nonhomogeneous equation, we have

$$\mathbf{\Psi}'(t)\mathbf{u}(t) + \mathbf{\Psi}(t)\mathbf{u}'(t) = A\mathbf{\Psi}(t)\mathbf{u}(t) + g(t).$$

Note that $\mathbf{\Psi}(t)$ is a fundamental matrix, thus we have

$$\mathbf{\Psi}'(t) = A\mathbf{\Psi}(t).$$

Hence, we have

$$\mathbf{\Psi}(t)\mathbf{u}'(t) = g(t) \implies \mathbf{u}'(t) = \mathbf{\Psi}^{-1}(t)g(t) \implies \mathbf{u}(t) = \int \mathbf{\Psi}^{-1}(t)g(t)\,dt + \mathbf{c}.$$

Here, $\mathbf{c}$ is an arbitrary constant vector. As a result, the general solution $\mathbf{x}$ has the form

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c}(t) + \mathbf{\Psi}(t)\int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)g(s)\,ds.$$

Here, t_0 is some given point of initial condition

$$\mathbf{x}(t_0) = \mathbf{x}^0.$$

Using the initial condition we have

$$\mathbf{\Psi}(t)\mathbf{c} = \mathbf{x}^0 \implies \mathbf{c} = \mathbf{\Psi}^{-1}(t)\mathbf{x}^0.$$

Example 7.9.3. Find a particular solution to the given system of equations:

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

using the method of variation of parameter.

Solution. We need first to compute a fundamental matrix for the associated homogeneous system

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x}.$$

To this aim, we compute the eigenvalues of the matrix by expanding the following determinant

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} = (1-\lambda)(-2-\lambda) - 4 = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3).$$

It implies that the matrix has two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -3$. For $\lambda_1 = 2$, one can find the corresponding eigenvector $\mathbf{v}^{(1)}$ such that

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\mathbf{v}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

thus we can choose

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Similarly, for $\lambda_2 = -3$, one can find the corresponding eigenvector $\mathbf{v}^{(2)}$ such that

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{v}^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

thus we can choose

$$\mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

Hence, a fundamental matrix $\mathbf{\Psi}(t)$ can be found as follows:

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{pmatrix} \implies \mathbf{\Psi}^{-1}(t) = \frac{1}{-5e^{-t}} \begin{pmatrix} -4e^{-3t} & -e^{-3t} \\ -e^{2t} & e^{2t} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4e^{-2t} & e^{-2t} \\ e^{3t} & -e^{3t} \end{pmatrix}.$$

Using the method of variation parameter, we can find out that

$$\mathbf{u}'(t) = \mathbf{\Psi}^{-1}(t)g(t) = \frac{1}{5} \begin{pmatrix} 4e^{-2t} & e^{-2t} \\ e^{3t} & -e^{3t} \end{pmatrix} \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4e^{-4t} - 2e^{-t} \\ e^{t} + 2e^{4t} \end{pmatrix}.$$

Integrating each of the component in $\mathbf{u}'(t)$, we get that

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix},$$

where

$$u_1(t) = \frac{1}{5} \int [4e^{-4t} - 2e^{-t}]\, dt = -\frac{1}{5}e^{-4t} + \frac{2}{5}e^{-t} + c_1,$$
$$u_2(t) = \frac{1}{5} \int [e^{t} + 2e^{4t}]\, dt = \frac{1}{5}e^{t} + \frac{1}{10}e^{4t} + c_2,$$

where c_1 and c_2 are arbitrary constants. As a result, we have

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -e^{-4t} + 2e^{-t} \\ e^{t} + e^{4t}/2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Therefore, the general solution of the nonhomogeneous system reads

$$\begin{aligned} \mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{u}(t) &= c_1\mathbf{v}^{(1)}e^{2t} + c_2\mathbf{v}^{(2)}e^{-3t} + \frac{1}{5} \begin{pmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{pmatrix} \begin{pmatrix} -e^{-4t} + 2e^{-t} \\ e^{t} + e^{4t}/2 \end{pmatrix} \\ &= c_1\mathbf{v}^{(1)}e^{2t} + c_2\mathbf{v}^{(2)}e^{-3t} + \frac{1}{5} \begin{pmatrix} -e^{-2t} + 2e^{t} + e^{-2t} + e^{t}/2 \\ -e^{-2t} + 2e^{t} - 4e^{-2t} - 2e^{t} \end{pmatrix} \\ &= c_1\mathbf{v}^{(1)}e^{2t} + c_2\mathbf{v}^{(2)}e^{-3t} + \frac{1}{2} \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{t}. \end{aligned}$$

□

7.10 Exercises

1. Transform the given initial-value problem into an initial-value problem for two first-order equations.
$$u'' + \frac{1}{4}u' + 4u = 2\cos(3t), \quad u(0) = 1, \quad u'(0) = -2.$$

2. Find all eigenvalues and eigenvectors of the given matrix
$$A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}.$$

3. Find the solution of the given initial-value problem.

   (a) $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$

   (b) $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$

   (c) $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

   (d) $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$

4. Find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = \mathbf{I}$ for the given first-order system:
$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}.$$

5. Find the general solution of the given system of equations.

   (a) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}e^t.$

   (b) $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}.$

Exercise

There are 9 questions in this assignment. Answer and hand in your work for those are not optional. No need to hand in your work for the optional problems. Please write down your name and UIN. The deadline is **11:59 pm (CDT), Dec 7 2022**. Problems 1. - 4. are for Chapter 5 and Problems 5. - 9. are for Chapter 7.

1. Seek power series solutions of the given differential equation about the given point x_0 and find the recurrence relation that the coefficients satisfy.

   (a) $y'' - xy' - y = 0$, $x_0 = 0$.

   (b) $2y'' + xy' + 3y = 0$, $x_0 = 0$.

   Solution. In the following, we always assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n, \quad \text{and} \quad y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

   (a) Plugging in the expressions of y, y', and y'', we obtain

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right] x^n - \sum_{n=0}^{\infty} (n+1)a_{n+1}x^{n+1} = 0.$$

   In the first summation on the left-hand side, we pull the term with index $n = 0$ out; and shift the index of the second summation by 1, we get

$$2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n - na_n\right] x^n = 0.$$

   The recursive relation is

$$a_2 = \frac{a_0}{2}, \quad a_{n+2} = \frac{(n+1)a_n}{(n+1)(n+2)} = \frac{a_n}{n+2} \quad \text{for } n = 1, 2, 3, \cdots.$$

   (b) Plugging in the expressions of y, y', and y'', we obtain

$$\sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2} - 3a_n\right] x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}x^{n+1} = 0.$$

   Similar to (a), we then get

$$4a_2 - 3a_0 + \sum_{n=1}^{\infty} \left[2(n+2)(n+1)a_{n+2} - 3a_n + na_n\right] x^n = 0.$$

   The recursive relation is

$$a_2 = \frac{3}{4}a_0, \quad a_{n+2} = \frac{(n-3)a_n}{2(n+1)(n+2)} \quad \text{for } n = 1, 2, 3, \cdots.$$

   □

2. Determine $y''(0)$, $y'''(0)$, and $y^{(4)}(0)$, where $y(x)$ is a solution of the given initial-value problem

$$y'' + xy' + y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution. Substitute $x = 0$ in the differential equation, we get

$$y''(0) + y(0) = 0 \implies y''(0) = -1.$$

Differentiating the equation (with respect to x), we get

$$y''' + xy'' + 2y' = 0 \implies y'''(0) = -2y'(0) = 0.$$

Differentiating the equation one more time, we get

$$y^{(4)} + xy''' + (x + 2)y'' = 0 \implies y^{(4)}(0) = -2y''(0) = -2.$$

□

3. **(Optional, no need to hand in)** Determine a lower bound for the radius of convergence of series solutions at each given point x_0 for the given differential equation

$$(x^2 - 2x - 3)y'' + xy' + 4y = 0,$$

where $x_0 = 4$, $x_0 = -4$, and $x_0 = 0$.

4. **(Optional, no need to hand in)** Consider the initial-value problem

$$y'' + (\sin x)y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Assume the solution is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

at $x = 0$. Find the first four nonzero terms in the series.

5. Transform the given initial-value problem into an initial-value problem for two first-order equations.

$$u'' + \frac{1}{4}u' + 4u = 2\cos(3t), \quad u(0) = 1, \quad u'(0) = -2.$$

Solution. Introducing $x_1 = u$ and $x_2 = u'$, we first get a relation between x_1 and x_2:

$$x_1' = x_2.$$

The original differential equation can be expressed in terms of x_1 and x_2:

$$u'' + \frac{1}{4}u' + 4u = 2\cos(3t) \implies x_2' = -4x_1 - \frac{1}{4}x_2 + 2\cos(3t).$$

Hence, we obtain the (nonhomogeneous) system of equations for x_1 and x_2

$$x_1' = x_2, \quad x_2' = -4x_1 - \frac{1}{4}x_2 + 2\cos(3t)$$

with initial conditions

$$x_1(0) = 1 \quad \text{and} \quad x_2(0) = -2.$$

□

6. **(Optional, no need to hand in)** Find all eigenvalues and eigenvectors of the given matrix

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}.$$

7. Find the solution of the given initial-value problem.

 (a) $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$

 Solution. We first find the eigenvalues and eigenvectors of the matrix. To this aim, we expand the following determinant of λ:

$$\begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda) + 3 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4).$$

 Hence, we obtain the eigenvalues of the matrix to be $\lambda_1 = 2$ and $\lambda_2 = 4$. For $\lambda_1 = 2$, we want to find the eigenvector $\mathbf{v}^{(1)}$ such that

$$\begin{pmatrix} 5-\lambda_1 & -1 \\ 3 & 1-\lambda_1 \end{pmatrix} \mathbf{v}^{(1)} = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \mathbf{v}^{(1)} = \mathbf{0}.$$

 Hence, we can choose

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

 Similarly, for $\lambda_2 = 4$, we want to find the eigenvector $\mathbf{v}^{(2)}$ such that

$$\begin{pmatrix} 5-\lambda_2 & -1 \\ 3 & 1-\lambda_2 \end{pmatrix} \mathbf{v}^{(2)} = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \mathbf{v}^{(2)} = \mathbf{0}.$$

 Hence, we can choose

$$\mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

 Hence, we obtain two linearly independent solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ to the system:

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} \quad \text{and} \quad \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.$$

 The general solution of the system is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$, where c_1 and c_2 are arbitrary and can be determined by the initial conditions. Making use of the initial conditions, we have the equations for c_1 and c_2:

$$c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \iff \begin{cases} c_1 + c_2 = 2, \\ 3c_1 + c_2 = -1. \end{cases}$$

 Hence, we obtain that

$$c_1 = -\frac{3}{2} \quad \text{and} \quad c_2 = \frac{7}{2}.$$

 The solution to the initial-value problem is

$$\mathbf{x}(t) = -\frac{3}{2}\mathbf{x}^{(1)} + \frac{7}{2}\mathbf{x}^{(2)} = \frac{1}{2} \begin{pmatrix} -3e^{2t} + 7e^{4t} \\ -9e^{2t} + 7e^{4t} \end{pmatrix}.$$

 □

 (b) **(Optional, no need to hand in)** $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$

 (c) $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

Solution. We first find the eigenvalues and eigenvectors of the matrix. To this aim, we expand the following determinant of λ:

$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda)+5 = \lambda^2+2\lambda+2 = (\lambda+1)^2+1.$$

Hence, we obtain the eigenvalues of the matrix to be $\lambda_1 = -1+i$ and $\lambda_2 = -1-i$. It is a complex-conjugated case. The (real-valued) general solution involves trigonometric terms. For $\lambda_1 = -1+i$, we want to find the eigenvector $\mathbf{v}^{(1)}$ such that

$$\begin{pmatrix} 1-\lambda_1 & -5 \\ 1 & -3-\lambda_1 \end{pmatrix} \mathbf{v}^{(1)} = \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \mathbf{v}^{(1)} = \mathbf{0}.$$

Hence, we can choose

$$\mathbf{v}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using the Euler's formula, we have

$$\mathbf{v}^{(1)} e^{\lambda_1 t} = e^{-t} \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] [\cos(t) + i\sin(t)].$$

We take $\mathbf{u}(t)$ and $\mathbf{v}(t)$ to be

$$\mathbf{u}(t) = e^{-t} \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cos(t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(t) \right],$$

$$\mathbf{v}(t) = e^{-t} \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t) \right].$$

The general solution of the system is $\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t)$, where c_1 and c_2 are arbitrary and can be determined by the initial conditions. Making use of the initial conditions, we have the equations for c_1 and c_2:

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \iff \begin{cases} 2c_1 + c_2 = 1, \\ c_1 = 1. \end{cases}$$

Hence, we obtain that

$$c_1 = 1 \quad \text{and} \quad c_2 = -1.$$

The solution to the initial-value problem is

$$\mathbf{x}(t) = \mathbf{u}(t) - \mathbf{v}(t) = e^{-t} \begin{pmatrix} \cos(t) - 3\sin(t) \\ \cos(t) - \sin(t) \end{pmatrix}.$$

□

(d) **(Optional, no need to hand in)** $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$

8. Find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = \mathbf{I}$ for the given first-order system:

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}.$$

Solution. If we write $\Phi(t)$ to be

$$\Phi(t) = \left[\mathbf{x}^{(1)}\ \mathbf{x}^{(2)}\right],$$

then $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ satisfy the system of equations with the following initial conditions

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively. Next, we find the eigenvalues and eigenvectors of the matrix. To this aim, we expand the following determinant of λ:

$$\begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (1+\lambda)^2 + 4.$$

Hence, we obtain the eigenvalues of the matrix to be $\lambda_1 = -1+2i$ and $\lambda_2 = -1-2i$. It is a complex-conjugated case. The (real-valued) general solution involves trigonometric terms. For $\lambda_1 = -1+2i$, we want to find the eigenvector $\mathbf{v}^{(1)}$ such that

$$\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \mathbf{v}^{(1)} = \mathbf{0}.$$

Hence, we can choose

$$\mathbf{v}^{(1)} = \begin{pmatrix} 2i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Using the Euler's formula, we have

$$\mathbf{v}^{(1)} e^{\lambda_1 t} = e^{-t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right] \left[\cos(2t) + i\sin(2t)\right].$$

We take $\mathbf{u}(t)$ and $\mathbf{v}(t)$ to be

$$\mathbf{u}(t) = e^{-t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2t) - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin(2t)\right],$$

$$\mathbf{v}(t) = e^{-t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2t) + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos(2t)\right].$$

The general solution of the system is $\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t)$, where c_1 and c_2 are arbitrary and can be determined by the initial conditions. We find $\mathbf{x}^{(1)}$ first. Making use of the initial conditions, we have the equations for c_1 and c_2:

$$c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \iff \begin{cases} 2c_2 = 1, \\ c_1 = 0. \end{cases}$$

Hence, we obtain that

$$c_1 = 0 \quad \text{and} \quad c_2 = \frac{1}{2}.$$

The solution to the initial-value problem is

$$\mathbf{x}^{(1)}(t) = \frac{1}{2}\mathbf{v}(t) = e^{-t} \begin{pmatrix} \cos(2t) \\ \sin(2t)/2 \end{pmatrix}.$$

Next, we find $\mathbf{x}^{(2)}$. Making use of the initial conditions, we have the equations for c_1 and c_2:

$$c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \iff \begin{cases} 2c_2 = 0, \\ c_1 = 1. \end{cases}$$

Hence, we obtain that

$$c_1 = 1 \quad \text{and} \quad c_2 = 0.$$

The solution to the initial-value problem is

$$\mathbf{x}^{(2)}(t) = \mathbf{u}(t) = e^{-t} \begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix}.$$

As a result, we have

$$\Phi(t) = e^{-t} \begin{pmatrix} \cos(2t) & -2\sin(2t) \\ \sin(2t)/2 & \cos(2t) \end{pmatrix}.$$

□

9. Find the general solution of the given system of equations.

(a) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t.$

(b) $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}.$

Solution. In the following, we use the method of variation of parameters to find the solution. We first find a fundamental matrix $\mathbf{\Psi}(t)$ of the associated homogeneous system. Then, we use the following formula to figure out $\mathbf{u}(t)$ such that

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{u}(t), \quad \text{where } \mathbf{u}(t) = \int \mathbf{\Psi}^{-1}(t) g(t)\, dt + \mathbf{c},$$

where $\mathbf{c}$ is a constant vector that contains the arbitrary constants c_1 and c_2 in the formula of general solution.

(a) We first find the eigenvalues and eigenvectors of the matrix. To this aim, we expand the following determinant of λ:

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3).$$

Hence, we obtain the eigenvalues of the matrix to be $\lambda_1 = -1$ and $\lambda_2 = 3$. For $\lambda_1 = -1$, we want to find the eigenvector $\mathbf{v}^{(1)}$ such that

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \mathbf{v}^{(1)} = \mathbf{0}.$$

Hence, we can choose

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Similarly, for $\lambda_2 = 3$, we want to find the eigenvector $\mathbf{v}^{(2)}$ such that

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{v}^{(2)} = \mathbf{0}.$$

Hence, we can choose

$$\mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

As a result, we obtain the fundamental matrix $\mathbf{\Psi}(t)$ as follows:

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix} \Longrightarrow \mathbf{\Psi}^{-1}(t) = \frac{1}{4e^{2t}} \begin{pmatrix} 2e^{3t} & -e^{3t} \\ 2e^{-t} & e^{-t} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2e^{t} & -e^{t} \\ 2e^{-3t} & e^{-3t} \end{pmatrix}.$$

Then, we can compute the vector $\mathbf{u}(t)$ as follows:

$$\mathbf{u}(t) = \int \mathbf{\Psi}^{-1}(t) g(t)\, dt + \mathbf{c} = \int \frac{1}{4} \begin{pmatrix} 2e^{t} & -e^{t} \\ 2e^{-3t} & e^{-3t} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}\, dt + \mathbf{c} = \frac{1}{4} \int \begin{pmatrix} 5e^{2t} \\ 3e^{-2t} \end{pmatrix} dt + \mathbf{c}.$$

Therefore, the vector $\mathbf{u}(t)$ is as follows:

$$\mathbf{u}(t) = \frac{1}{8} \begin{pmatrix} 5e^{2t} \\ -3e^{-2t} \end{pmatrix} + \mathbf{c}.$$

The general solution is

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{u}(t) = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^{t} + c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}.$$

(b) We first find the eigenvalues and eigenvectors of the matrix. To this aim, we expand the following determinant of λ:

$$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 5 = \lambda^2 + 1.$$

Hence, we obtain the eigenvalues of the matrix to be $\lambda_1 = i$ and $\lambda_2 = -i$. It is a complex-conjugated case. The (real-valued) general solution involves trigonometric terms. For $\lambda_1 = i$, we want to find the eigenvector $\mathbf{v}^{(1)}$ such that

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \mathbf{v}^{(1)} = \mathbf{0}.$$

Hence, we can choose

$$\mathbf{v}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using the Euler's formula, we have

$$\mathbf{v}^{(1)} e^{\lambda_1 t} = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] [\cos(t) + i \sin(t)].$$

Then, we obtain the fundamental matrix $\mathbf{\Psi}(t)$ to be

$$\mathbf{\Psi}(t) = \left(\mathbf{x}^{(1)}\ \mathbf{x}^{(2)} \right),$$

where

$$\mathbf{x}^{(1)}(t) = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cos(t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(t) \right],$$

$$\mathbf{x}^{(2)}(t) = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t) \right].$$

As a result, we have

$$\mathbf{\Psi}(t) = \begin{pmatrix} 2\cos(t) - \sin(t) & 2\sin(t) + \cos(t) \\ \cos(t) & \sin(t) \end{pmatrix} \implies \mathbf{\Psi}^{-1}(t) = \frac{1}{-1} \begin{pmatrix} \sin(t) & -2\sin(t) - \cos(t) \\ -\cos(t) & 2\cos(t) - \sin(t) \end{pmatrix}.$$

That is,

$$\mathbf{\Psi}^{-1}(t) = \begin{pmatrix} -\sin(t) & 2\sin(t) + \cos(t) \\ \cos(t) & -2\cos(t) + \sin(t) \end{pmatrix}.$$

Then, we can compute the vector $\mathbf{u}(t)$ as follows:

$$\mathbf{u}(t) = \int \mathbf{\Psi}^{-1}(t) g(t)\, dt + \mathbf{c} = \int \begin{pmatrix} -\sin(t) & 2\sin(t) + \cos(t) \\ \cos(t) & -2\cos(t) + \sin(t) \end{pmatrix} \begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix} dt + \mathbf{c}.$$

That is,

$$\begin{aligned} \mathbf{u}(t) &= \int \begin{pmatrix} 2\sin(t)\cos(t) + 2\sin^2(t) \\ -\cos^2(t) - 2\sin(t)\cos(t) + \sin^2(t) \end{pmatrix} dt + \mathbf{c} = \int \begin{pmatrix} \sin(2t) + 2\sin^2(t) \\ -\cos(2t) - \sin(2t) \end{pmatrix} dt + \mathbf{c} \\ &= \int \begin{pmatrix} \sin(2t) + 1 - \cos(2t) \\ -\cos(2t) - \sin(2t) \end{pmatrix} dt + \mathbf{c} = \frac{1}{2} \begin{pmatrix} -\cos(2t) + 2t - \sin(2t) \\ -\sin(2t) + \cos(2t) \end{pmatrix} + \mathbf{c}. \end{aligned}$$

The general solution is

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{u}(t).$$

□

Chapter 8

Numerical Methods

8.1 The Euler or Tangent Line Method

8.2 Improvements on the Euler Method

8.3 The Runge-Kutta Method

Chapter 9

Nonlinear Differential Equations and Stability

9.1 The Phase Plane: Linear Systems

9.2 Autonomous Systems and Stability

9.3 Locally Linear Systems

Chapter 10

Partial Differential Equations and Fourier Series

Chapter 11

Boundary Value Problems and Sturm-Liouville Theory

Appendix A

Python Lectures